TRAVELING WAVES FOR A THIN LIQUID FILM WITH SURFACTANT ON AN INCLINED PLANE

VAHAGN MANUKIAN AND STEPHEN SCHECTER

Abstract. We show the existence of traveling wave solutions for a lubrication model of surfactant-driven flow of a thin liquid film down an inclined plane, in various parameter regimes. Our arguments use geometric singular perturbation theory.

1. Introduction

We study the flow of a thin liquid film down an inclined plane in the presence of a surfactant which is insoluble and therefore remains on the liquid surface. The governing system of partial differential equations includes an equation for the fluid height and an equation for the surfactant concentration. After an initial normalization, the system contains three parameters, which are proportional to deviation of the inclined plane from the perpendicular, the capillary number of the liquid, and the diffusion constant of the surfactant. The model includes the Marangoni force, a tangential force at the liquid surface due to spatial variation in the surfactant concentration. The reader who is unfamiliar with the Marangoni force can observe its effect by placing a small piece of paper in a pan of water and adding a drop of dishwashing liquid.

The system of partial differential equations in one space dimension is

$$h_t - \frac{1}{2}\left(h^2\Gamma_x\right)_x + \frac{\alpha}{3}\left(h^3\right)_x = -\frac{C}{3}\left(h^3 h_{xxx}\right)_x + \frac{\beta}{3}\left(h^3 h_x\right)_x, \tag{1.1}$$

$$\Gamma_t - \left(h\Gamma\Gamma_x\right)_x + \frac{\alpha}{2}\left(h^2\Gamma\right)_x = -\frac{C}{2}\left(h^2 h_{xxx}\Gamma\right)_x + \frac{\beta}{2}\left(h^2 h_x\Gamma\right)_x + D\Gamma_{xx}. \tag{1.2}$$

In these equations, x increases as one descends the inclined plane; h is the height of the liquid film; Γ is concentration of the surfactant on the liquid surface; α (respectively β) is proportional to the sine (respectively cosine) of the smaller angle that the inclined plane makes with the horizontal; C is proportional to the capillary number of the liquid; and D is proportional to the diffusion constant of the surfactant. We assume α and h are positive, so the plane is not horizontal and is not dry anywhere; Γ, β, C, and D are nonnegative. After an initial normalization, α can be made equal to 1.

Our work is motivated by the papers [10] and [11], where the reader will find discussion of the physical background and references to the literature. The focus of these papers is the existence of traveling wave solutions for which $h(-\infty) = h_L > h_R = h(\infty)$ are given

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and $\int_{-\infty}^{\infty} \Gamma(x)\,dx$ is finite. The latter condition means that the total amount of surfactant is finite; the integral is an invariant of the system. For certain values of the parameters, h_L, and h_R, the authors of these papers find that there is a one-parameter family of such traveling waves, parameterized, roughly speaking, by $\int_{-\infty}^{\infty} \Gamma(x)\,dx$. In other words, given the amount of surfactant, there is a traveling wave.

The authors of [10] and [11] consider cases in which at least one of the parameters C, D, and β is zero. Our interest in the present paper is the case in which all three of these parameters are positive. Another contrast with [10] and [11] is that we make extensive use of geometric singular perturbation theory [6, 7, 8].

For the same range of h_L and h_R considered in [10] and [11] we show, in certain parts of parameter space, the existence of a one-parameter family of traveling waves, and describe the outstanding features of these waves. In other parts of parameter space we elucidate the underlying geometry but are not able to provide existence proofs.

The plan of the paper is as follows. We begin, in Section 2, by deriving three normalizations of the system (1.1)–(1.2), and we describe six regions of parameter space that we will study in some detail; see Figure 2.1. In Section 3 we derive the traveling wave system as a first-order system in $\mathbb{R}^4$. In Section 6 we examine equilibria and invariant spaces for the traveling wave system in $\mathbb{R}^4$. The traveling waves we seek correspond to solutions of the traveling wave system that connect two equilibria. The equilibria have unstable and stable manifolds of dimensions 3 and 3 respectively, so if they meet transversally, they do so in a two-dimensional manifold, which represents a one-parameter family of traveling waves.

In Section 4, we show numerical simulations of some traveling waves, in order to familiarize the reader with their typical features.

In Section 5 we derive some preliminary results. One is a compactification of the traveling wave system at $\Gamma = \infty$ that is convenient for studying traveling waves with large Γ, i.e., large concentrations of surfactant. Two other preliminary results give existence and transversality of connecting orbits for a class of first-order systems in $\mathbb{R}^3$. These systems correspond to the third-order equation $\dddot{h} - B\dot{h} + F(h) = 0$ with $B \geq 0$, $F(h) = 0$ at two points, and $F(h) < 0$ if and only if h is between those points. There is a literature on existence of connecting orbits for systems in this class, which we review. The transversality result seems to be new. It states that if F has a unique critical point and, along the connecting orbit, $h(t)$ passes through that point only once, then the connecting orbit represents a transversal intersection of the unstable and stable manifolds of two equilibria.

In Section 7 we show that for any positive parameter values, connecting orbits of the traveling wave system with Γ small exist, provided the transversality condition just mentioned holds within the three-dimensional invariant space $\Gamma = 0$.

In Sections 8–13 we investigate the six regions of parameter space in some detail. For each of the six regions, we work with one of the three normalizations of (1.1)–(1.2). We fix h_L and h_R in the range considered in [10] and [11] and study the existence of connecting orbits.

In each region at least one normalized parameter is small. Our method is always to allow the small normalized parameters to approach 0, possibly with some rescalings, and look at the limit. In Region 1 we explain some of the geometry but are not able to prove anything. In Regions 2 and 4 there is a two-dimensional normally hyperbolic invariant manifold on which connecting orbits with $\max \Gamma$ arbitrary exist. In Region 3, if a transversality condition holds, then again connecting orbits with $\max \Gamma$ arbitrary exist, but they do not stay in a two-dimensional normally hyperbolic invariant manifold.

Regions 5 and 6 are the most interesting from the point of view of geometric singular perturbation theory. In both regions a two-dimensional invariant manifold loses normal hyperbolicity when Γ and a parameter become 0. In this situation Γ and the parameter must be simultaneously rescaled. In geometric singular perturbation theory this procedure is called blowing-up [5, 9]. Typically one obtains a system on a neighborhood of a sphere, which must be studied in several coordinate patches. In our situations, however, we are able to choose coordinates in which the connecting orbits lie in a single coordinate patch, which we believe makes the analysis easier to follow.

In Regions 5 and 6, we do not show that for fixed parameter values, connecting orbits with $\max\Gamma$ arbitrary exist. Instead, in Region 5 (respectively Region 6), connecting orbits with $\max\Gamma \le \eta^{-1}$ (respectively $\eta \le \max\Gamma \le \eta^{-1}$) are proved to exist, where $\eta \to 0$ as a parameter goes to 0.

In Region 5 the connecting orbits again lie in a two-dimensional invariant manifold, but in the limit this manifold becomes two planes at right angles to each other, with normally hyperbolic equilibria along the line of intersection. Tracking solutions as they pass near such a line of equilibria requires what the second author has called a "corner lemma" [16]. We prove such a lemma appropriate to the present situation in Section 12.3.

In Region 6 the connecting orbits begin and end in the three-dimensional unstable and stable manifolds of the equilibria, and their middle portions lie near (not in) two-dimensional invariant manifolds like those for Region 5. However, the line of equilibria in the limit is not normally hyperbolic. Thus two additional complications are introduced. A corner lemma appropriate to this situation is proved in Section 15.

The proofs of the corner lemmas follow an outline based on Deng's Lemma that was recently used to prove a general exchange lemma [17].

Complex eigenvalues at an equilibrium of the traveling wave equation correspond to traveling waves that oscillate at the end that corresponds to that equilibrium. They occur in Region 1 and 3. Complex eigenvalues are discussed in Section 6.

2. Partial differential equations and normalizations

The scaling

$$\tilde{x} = \alpha x, \quad \tilde{t} = \alpha^2 t \tag{2.1}$$

converts (1.1)–(1.2) to the following equivalent system, in which we have dropped the tildes over the scaled variables:

$$h_t - \frac{1}{2}\left(h^2\Gamma_x\right)_x + \frac{1}{3}\left(h^3\right)_x = -\frac{\tilde{C}}{3}\left(h^3 h_{xxx}\right)_x + \frac{\tilde{\beta}}{3}\left(h^3 h_x\right)_x, \tag{2.2}$$

$$\Gamma_t - \left(h\Gamma\Gamma_x\right)_x + \frac{1}{2}\left(h^2\Gamma\right)_x = -\frac{\tilde{C}}{2}\left(h^2 h_{xxx}\Gamma\right)_x + \frac{\tilde{\beta}}{2}\left(h^2 h_x\Gamma\right)_x + \tilde{D}\Gamma_{xx}. \tag{2.3}$$

In (2.2)–(2.3), $\tilde{C} = \alpha^2 C$, $\tilde{D} = D$, and $\tilde{\beta} = \beta$. Notice that α has become 1.

In (2.2)–(2.3), the system with parameters $(\tilde{C}, \tilde{D}, \tilde{\beta})$ is considered equivalent to the system with parameters $(\hat{C}, \hat{D}, \hat{\beta})$ if the first can be converted to the second by scaling the variables x, t, and Γ; we avoid scaling h in order to keep the values of h_L and h_R fixed. The scaling is necessarily of the form

$$\hat{x} = kx, \quad \hat{t} = kt, \quad \hat{\Gamma} = k\Gamma, \tag{2.4}$$

with $k > 0$. Then $\hat{C} = k^3\tilde{C}$, $\hat{D} = k\tilde{D}$, and $\hat{\beta} = k\tilde{\beta}$. The equivalence class of each point other than the origin in $\{(\tilde{C}, \tilde{D}, \tilde{\beta}) : \tilde{C} \geq 0, \tilde{D} \geq 0, \tilde{\beta} \geq 0\}$ is a curve. See Figure 2.1.

By appropriate choice of k we can make $\tilde{C}$, $\tilde{D}$, or $\tilde{\beta}$ equal to 1. We thus obtain three two-parameter normalizations of (1.1)–(1.2). They are system (2.2)–(2.3) with the parameters

(N1) $\tilde{C}_1 = 1$, $\tilde{D}_1 = \alpha^{-\frac{2}{3}}C^{-\frac{1}{3}}D$, and $\tilde{\beta}_1 = \alpha^{-\frac{2}{3}}C^{-\frac{1}{3}}\beta$; or
(N2) $\tilde{C}_2 = \alpha^2 C D^{-3}$, $\tilde{D}_2 = 1$, $\tilde{\beta}_2 = D^{-1}\beta$; or
(N3) $\tilde{C}_3 = \alpha^2 C\beta^{-3}$, $\tilde{D}_3 = D\beta^{-1}$, and $\tilde{\beta}_3 = 1$.

Each two-parameter normal form can be thought of as a local cross section to the equivalence classes in $\tilde{C}\tilde{D}\tilde{\beta}$-space. See Figure 2.1.

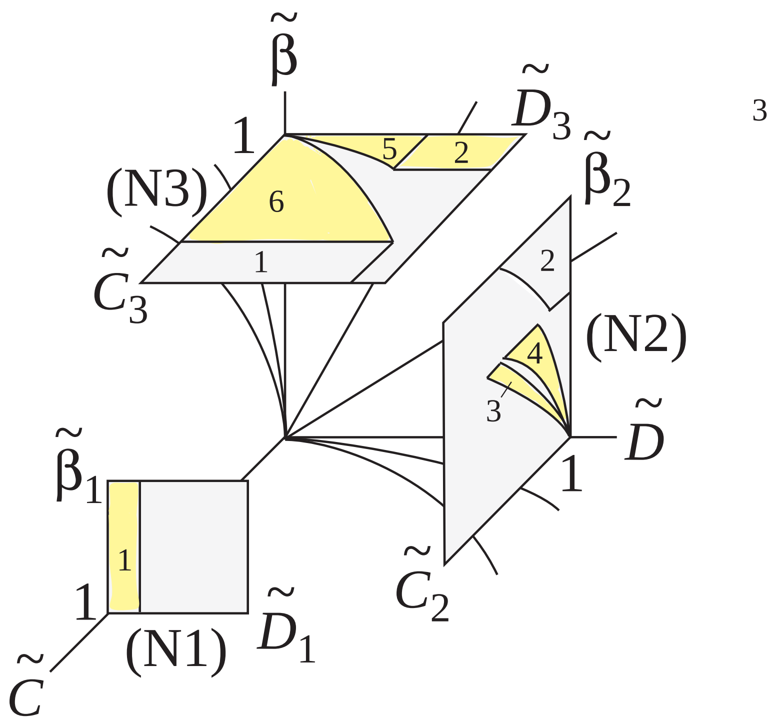


FIGURE 2.1. The equivalence classes include lines through the origin in $\tilde{D}\tilde{\beta}$-space; curves $\tilde{C}$ = constant$\cdot\tilde{D}^3$ in $\tilde{C}\tilde{D}$-space; and curves $\tilde{C}$ = constant$\cdot\tilde{\beta}^3$ in $\tilde{C}\tilde{\beta}$-space. The normalized coordinate systems (N1), (N2), and (N3) are shown, as are Regions 1–6 described below. Each region is colored in the coordinate system in which it is analyzed.

The three normal forms are related as follows:

$$\tilde{C}_2 = \tilde{D}_1^{-3}, \quad \tilde{\beta}_2 = \tilde{D}_1^{-1}\tilde{\beta}_1; \qquad \tilde{D}_1 = \tilde{C}_2^{-\frac{1}{3}}, \quad \tilde{\beta}_1 = \tilde{C}_2^{-\frac{1}{3}}\tilde{\beta}_2; \tag{2.5}$$

$$\tilde{C}_3 = \tilde{\beta}_2^{-3}\tilde{C}_2, \quad \tilde{D}_3 = \tilde{\beta}_2^{-1}; \qquad \tilde{C}_2 = \tilde{C}_3\tilde{D}_3^{-3}, \quad \tilde{\beta}_2 = \tilde{D}_3^{-1}; \tag{2.6}$$

$$\tilde{C}_3 = \tilde{\beta}_1^{-3}, \quad \tilde{D}_3 = \tilde{\beta}_1^{-1}\tilde{D}_1; \qquad \tilde{D}_1 = \tilde{C}_3^{-\frac{1}{3}}\tilde{D}_3, \quad \tilde{\beta}_1 = \tilde{C}_3^{-\frac{1}{3}}. \tag{2.7}$$

Our goal is to study the existence and structure of traveling waves of (1.1)–(1.2) of the form

$$(h(\xi), \Gamma(\xi)), \quad \xi = x - st, \quad h(-\infty) = h_L, \quad h(\infty) = h_R, \quad \Gamma(\pm\infty) = 0. \tag{2.8}$$

We have replaced the condition that $\int_{-\infty}^{\infty} \Gamma(\xi)\, d\xi$ be finite with the condition $\Gamma(\pm\infty) = 0$; the latter implies the former if $\Gamma(\xi)$ approaches 0 exponentially as $\xi \to \pm\infty$, which will be the case.

We are concerned only with C, D, and β all positive. However, we are especially interested in the limits $C \to 0$, $D \to 0$, and $\beta \to 0$, as well as allowing more than one of these parameters to approach 0 simultaneously. We shall study the cases in which one or two parameters approach 0 by allowing one or two normalized parameters to approach 0 in the system (2.2)–(2.3) with parameters (N1), (N2), or (N3). By reinterpreting the results one of course obtains considerable information about the limit in which all three of C, D, β approach 0.

We shall study the following six regions in some detail. We describe each region in only the system of normalized parameters in which we shall study it. Formulas (2.5)–(2.7) and Figure 2.1 can be used to find the equivalent regions in the other systems. In the description of each region, $\eta_i > 0$ and $\delta_i > 0$ are small; if both are used, the choice of δ_i depends on η_i.

- Region 1. $\tilde{C}_1 = 1$, $0 < \tilde{D}_1 \leq \delta_1$, $0 < \tilde{\beta}_1 \leq \eta_1^{-1}$.
- Region 2. $0 < \tilde{C}_3 \leq \delta_2$, $\eta_2 \leq \tilde{D}_3 \leq \eta_2^{-1}$, $\tilde{\beta}_3 = 1$.
- Region 3. $\eta_3 \tilde{\beta}_2^3 \leq \tilde{C}_2 \leq \eta_3^{-1} \tilde{\beta}_2^3$, $\tilde{D}_2 = 1$, $0 < \tilde{\beta}_2 \leq \delta_3$.
- Region 4. $\eta_4 \tilde{\beta}_2^5 \leq \tilde{C}_2 \leq \eta_4^{-1} \tilde{\beta}_2^5$, $\tilde{D}_2 = 1$, $0 < \tilde{\beta}_2 \leq \delta_4$.
- Region 5. $0 < \tilde{C}_3 \leq \delta_5 \tilde{D}_3^2$, $0 < \tilde{D}_3 \leq \delta_5$, $\tilde{\beta}_3 = 1$.
- Region 6. $0 < \tilde{C}_3 \leq \delta_6$, $0 < \tilde{D}_3 \leq \delta_6 \tilde{C}_3^{\frac{1}{2}}$, $\tilde{\beta}_3 = 1$.

The numbers η_i and δ_i are chosen separately for each region, so, depending on these choices, many of the regions may overlap. Figure 2.1 shows the regions for one way of choosing the numbers η_i and δ_i. Regions 2 and 5 can overlap, as can Regions 1 and 6; these pairs of regions are shown as adjacent.

3. Traveling wave system

In the normalized system (2.2)–(2.3), we look for a solution of the form (2.8). After integration we obtain the system of ODEs

$$-sh - \frac{1}{2}h^2\Gamma' + \frac{1}{3}h^3 = -\frac{\tilde{C}}{3}h^3 h''' + \frac{\tilde{\beta}}{3}h^3 h' + K_1, \tag{3.1}$$

$$-s\Gamma - h\Gamma\Gamma' + \frac{1}{2}h^2\Gamma = -\frac{\tilde{C}}{2}h^2 h'''\Gamma + \frac{\tilde{\beta}}{2}h^2 h'\Gamma + \tilde{D}\Gamma', \tag{3.2}$$

where

$$s = \frac{1}{3}(h_L^2 + h_L h_R + h_R^2), \quad K_1 = -\frac{1}{3}h_L h_R (h_L + h_R).$$

We now rewrite (3.1) and (3.2) as a first-order system. We first rewrite (3.1) and (3.2) as

$$\tilde{\beta}h' - \tilde{C}h''' = \frac{1}{h^3}\left(h^3 - 3sh - 3K_1 - \frac{3}{2}h^2\Gamma'\right), \tag{3.3}$$

$$\tilde{\beta}h' - \tilde{C}h''' = \frac{2}{h^2\Gamma}\left(\frac{1}{2}h^2\Gamma - s\Gamma - h\Gamma\Gamma' - \tilde{D}\Gamma'\right). \tag{3.4}$$

We equate the right-hand sides of (3.3) and (3.4), and solve for Γ':

$$\Gamma' = \frac{2\Gamma}{h}\frac{sh+3K_1}{h\Gamma+4\tilde{D}}. \tag{3.5}$$

Next we substitute (3.5) into (3.3):

$$\tilde{\beta}h' - \tilde{C}h''' = \frac{1}{h^3}\left(h^3 - 3sh - 3K_1 - 3h\Gamma\frac{sh+3K_1}{h\Gamma+4\tilde{D}}\right). \tag{3.6}$$

Writing (3.6), (3.5) as a first-order system, we obtain

$$h' = u, \tag{3.7}$$

$$u' = v, \tag{3.8}$$

$$\tilde{C}v' = \tilde{\beta}u - \frac{1}{h^3}\left(h^3 - 3sh - 3K_1 - 3h\Gamma\frac{sh+3K_1}{h\Gamma+4\tilde{D}}\right), \tag{3.9}$$

$$\Gamma' = \frac{2\Gamma}{h}\frac{sh+3K_1}{h\Gamma+4\tilde{D}}. \tag{3.10}$$

To avoid the possibility of dividing by zero in a limit, we rescale by multiplying the right hand side of (3.7)–(3.10) by $h\Gamma + 4\tilde{D}$. Let

$$H(h) = \frac{1}{h^3}\left(h^3 - 3sh - 3K_1\right), \quad Q(h) = \frac{2}{h}(sh+3K_1),$$
$$P(h) = H(h) - \frac{3}{2h^2}Q(h) = \frac{1}{h^3}\left(h^3 - 6sh - 12K_1\right)$$

We obtain the traveling wave system

$$\dot{h} = (h\Gamma + 4\tilde{D})u, \tag{3.11}$$

$$\dot{u} = (h\Gamma + 4\tilde{D})v, \tag{3.12}$$

$$\tilde{C}\dot{v} = h\Gamma\left(\tilde{\beta}u - P(h)\right) + 4\tilde{D}\left(\tilde{\beta}u - H(h)\right), \tag{3.13}$$

$$\dot{\Gamma} = \Gamma Q(h). \tag{3.14}$$

Note that s and K_1 have been absorbed into P, Q, and H. *Thus h_L and h_R have been fixed; however, $\tilde{C}$, $\tilde{D}$, and $\tilde{\beta}$ are parameters.* We consider this system with $h > 0$, u and v arbitrary, $\Gamma \geq 0$.

We shall assume

$$0 < h_R < h_L < \frac{1}{2}(\sqrt{3}-1)h_R. \tag{3.15}$$

Then by [11] there are numbers h_1, h_*, and h_2, with

$$0 < h_R < h_2 < h_* < h_L < h_1,$$

such that

$$H(h) = \frac{1}{h^3}(h-h_L)(h-h_R)(h+h_L+h_R), \quad Q(h) = \frac{2s}{h}(h-h_*),$$
$$P(h) = \frac{1}{h^3}(h-h_1)(h-h_2)(h+h_1+h_2).$$

See Figure 3.1.

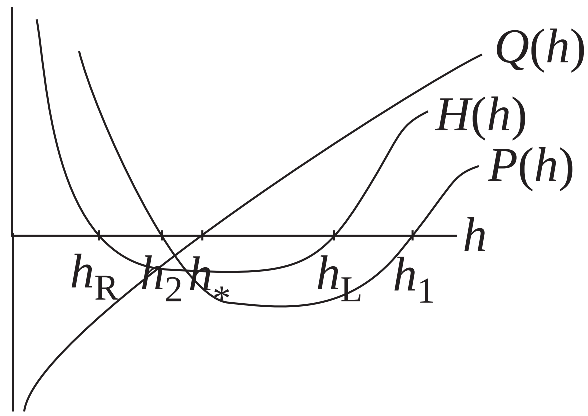


FIGURE 3.1

We are interested in solutions of the traveling wave system (3.11)–(3.14) that satisfy the boundary conditions

$$(h, u, v, \Gamma)(-\infty) = (h_L, 0, 0, 0), \quad (h, u, v, \Gamma)(\infty) = (h_R, 0, 0, 0). \tag{3.16}$$

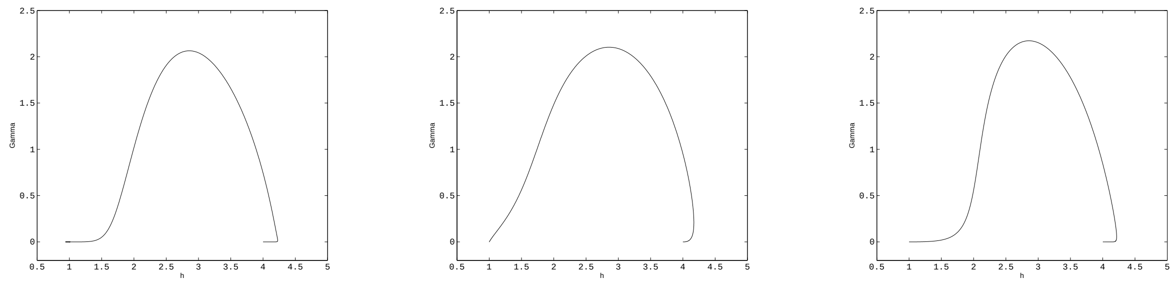
These solutions correspond to traveling wave solutions of the system of PDEs (2.2)–(2.3) of the form (2.8).

4. Numerical results

We consider the traveling wave system (3.11)–(3.14), with the boundary conditions (3.16). Thus we want solutions that lie in the unstable manifold of $(h_L, 0, 0, 0)$ and the stable manifold of $(h_R, 0, 0, 0)$. Denote the linear approximations of these manifolds by $E^u(h_L, 0, 0, 0)$ and $E^s(h_R, 0, 0, 0)$; their equations are readily computable. In order to approximate solutions numerically, we choose times $T_1 < T_2$ and search for solutions that lie in $E^u(h_L, 0, 0, 0)$ at time T_1 and in $E^s(h_R, 0, 0, 0)$ at time T_2. The error due to this approximation decays exponentially as T_1 and T_2 increase; see [1] for the method of analysis. The numbers $\tilde{C}$, $\tilde{D}$, $\tilde{\beta}$, h_L, h_R, $\int_{T_1}^{T_2} \Gamma(t)\,dt$, T_1, and T_2 are regarded as parameters.

We set $\tilde{C} = \tilde{D} = \tilde{\beta} = 1$, $h_L = 4$, $h_R = 1$, and $\int_{T_1}^{T_2} \Gamma(t)\,dt = 0$. The last choice implies that $\Gamma \equiv 0$, so the equation (3.11) can temporarily be dropped. Using xpp (available from http://www.math.pitt.edu/ bard/xpp/xpp.html) we found pictorially a solution of (3.12)–(3.14) with $\Gamma = 0$ that approximately connects $(4, 0, 0)$ to $(1, 0, 0)$. The time interval on which the solution is defined gives T_1 and T_2. We now have an approximate solution of our boundary value problem for certain values of the parameters. Using AUTO (available from http://indy.cs.concordia.ca/auto) we can improve this solution and, in principle, continue it as any of the parameters vary.

Figure 4.1 shows three computed traveling waves with $h_L = 4$, $h_R = 1$, and $\max \Gamma$ approximately 1.2; ; the waves are shown as parameterized curves in $huv\Gamma$-space projected into the $h\Gamma$-plane. In Figure 4.1(a) the parameters are in or close to Region 1. The solution oscillates near $(h, u, v, \Gamma) = (1, 0, 0, 0)$. This can be seen more clearly when the solution is projected onto hu-space instead of $h\Gamma$-space; see Figure 4.2. In Figure 4.1(b) the parameters are in or close to Region 2. The projected solution approaches $(4, 0)$ along the Γ-axis but approaches $(1, 0)$ from an oblique direction; there is no oscillation. In Figure 4.1(c) the parameters are

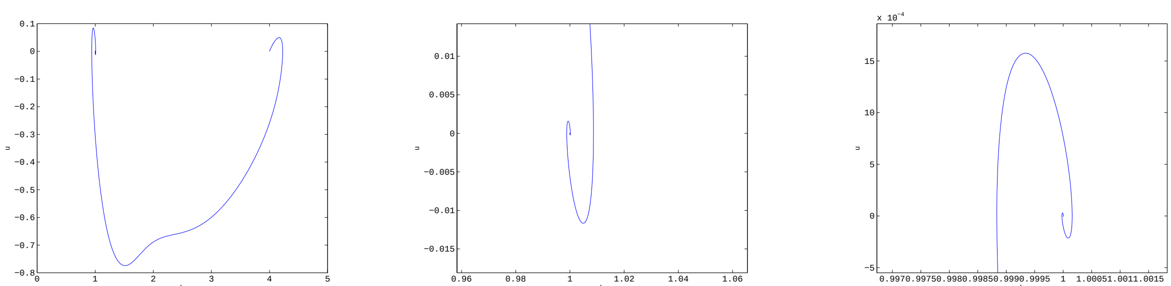


(a) (b) (c)

FIGURE 4.1. Three computed traveling waves with $h_L = 4$, $h_R = 1$, and $\max \Gamma$ approximately 1.2; the waves are shown as curves in the $h\Gamma$-plane that connect $(4, 0)$ to $(1, 0)$. (a) $(\tilde{C}, \tilde{D}, \tilde{\beta}) = (1.0, 0.361644, 1.0)$. (b) $(\tilde{C}, \tilde{D}, \tilde{\beta}) = (0.000217946, 1.0, 1.0)$. (c) $(\tilde{C}, \tilde{D}, \tilde{\beta}) = (0.0000671477, 0.207638, 1.0)$.

in or close to Region 5. The projected solution approaches both $(4, 0)$ and $(1, 0)$ along the Γ-axis; there is no oscillation.

We refer to the portions of these curves that are near and approximately parallel to the Γ-axis as "feet." Figures 4.1(a) and (c) have two feet; Figure 4.1(b) has one. If we vary the parameters in Figure 4.1(b) a little, the projected solution continues to approach $(1, 0)$ obliquely, so there is still only one foot.



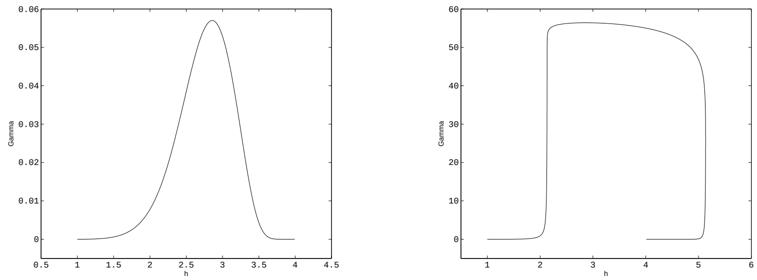
(a) (b) (c)

FIGURE 4.2. (a) The solution shown in Figure 4.1(a) projected onto hu-space instead of $h\Gamma$-space. (b) Zoom into a small rectangle around $(h, u) = (1, 0)$. (c) Zoom into a smaller rectangle around $(h, u) = (1, 0)$.

Figure 4.3 shows the effect of changing $\max \Gamma$. The parameter values are those of Figure 4.1(c). For $\max \Gamma$ small the foot near $(1, 0)$ points out. For $\max \Gamma$ large two portions of the curve are approximately vertical. They are near $h = h_1$ and $h = h_2$ defined earlier.

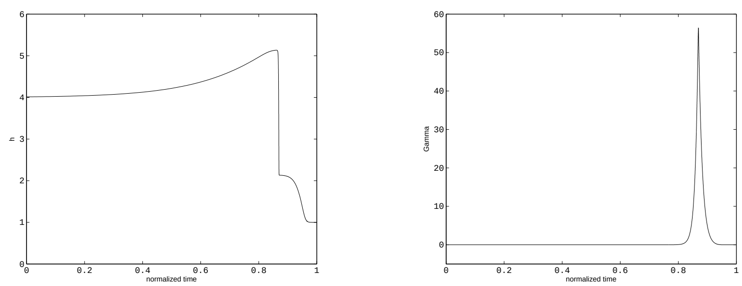
For the traveling wave shown in Figure 4.3(b) Figure 4.4 shows h and Γ as functions of normalized time. (AUTO normalizes the time interval $[T_1, T_2]$ to $[0, 1]$.) The h-component of the wave exhibits a clear four-level structure that is not as apparent when $\max \Gamma$ is small; the steps are at $h = h_L$, $h \sim h_1$, $\sim h_2$, and $h = h_R$ Virtually the entire change in Γ occurs close the time period when h falls from near h_1 to near h_2.

For more plots of traveling waves see [11].



(a) (b)

FIGURE 4.3. Traveling waves shown as curves in the $h\Gamma$-plane. The parameter values are those of Figure 4.1(c). (a) Solution with $\max \Gamma$ approximately 57. (b) Solution with $\max \Gamma$ approximately 60.0.



(a) (b)

FIGURE 4.4. The traveling wave of Figure 4.3(b); h and Γ are shown as functions of normalized time. (AUTO normalizes the time interval in a boundary value problem to $[0, 1]$.)

5. Preliminaries

5.1. **Compactification of the Γ-interval.** We wish to consider traveling waves of all sizes, so that Γ may be large. In order to deal conveniently with the interval $0 \le \Gamma < \infty$, in (3.11)–(3.14) we make the change of variables

$$\Gamma = \frac{\gamma}{1-\gamma}, \tag{5.1}$$

so that the interval $0 \le \Gamma < \infty$ corresponds to $0 \le \gamma < 1$. After multiplying by $1-\gamma$, we obtain

$$\dot{h} = (h\gamma + 4\tilde{D}(1-\gamma))u, \tag{5.2}$$

$$\dot{u} = (h\gamma + 4\tilde{D}(1-\gamma))v, \tag{5.3}$$

$$\tilde{C}\dot{v} = h\gamma \left(\tilde{\beta}u - P(h)\right) + 4\tilde{D}(1-\gamma)\left(\tilde{\beta}u - H(h)\right), \tag{5.4}$$

$$\dot{\gamma} = \gamma(1-\gamma)^2 Q(h). \tag{5.5}$$

We consider this system on $h > 0$, u and v arbitrary, $0 \le \gamma \le 1$. Note that the infinite Γ-interval has been compactified.

5.2. **Linear combinations of hP and H.** The following proposition will be useful. Most of it is proved in [11].

Proposition 5.1. *Let $F(a, b, h) = ahP(h) + bH(h)$. If $a \ge 0$, $b \ge 0$, and $(a, b) \ne (0, 0)$, then:*

(1) *For $0 < h < h_R$ (respectively $h_1 < h < h_L$, respectively $h_2 < h < \infty$), $F(a, b, h)$ is positive (respectively negative, respectively positive).*
(2) *The equation $F(a, b, h) = 0$ has exactly two solutions in $h > 0$, which we denote $h_2(a, b) < h_1(a, b)$. We have $h_R \le h_2(a, b) \le h_2$ and $h_L \le h_1(a, b) \le h_1$.*
(3) *$h_2(0, b) = h_R$, $h_2(a, 0) = h_2$, $h_1(0, b) = h_L$, and $h_1(a, 0) = h_1$.*
(4) *$\frac{\partial F}{\partial h}(a, b, h_2(a, b)) < 0$ and $\frac{\partial F}{\partial h}(a, b, h_1(a, b)) > 0$.*
(5) *$\int_0^{h_1(a,b)} F(a, b, h)\, dh > 0$ and $\int_{h_2(a,b)}^{\infty} F(a, b, h)\, dh > 0$.*
(6) *The equation $\frac{\partial F}{\partial h}(a, b, h) = 0$ has a unique solution in $h > 0$.*

Proof. (1) is a consequence of the signs of P and H on the given intervals; see Figure 3.1.

For $b = 0$ or $a = 0$, (2)–(6) follow from the formulas for P and H. We therefore assume $a > 0$ and $b > 0$.

Write $F(a, b, h) = \frac{a}{h^3}(h^4 + a_3h^3 + a_2h^2 + a_1h + a_0) = \frac{a}{h^3}(h - \mu_1)(h - \mu_2)(h - \mu_3)(h - \mu_4)$. For $a > 0$ and $b > 0$, we see from the formulas for P and H that $a_3 = -(\mu_1 + \mu_2 + \mu_3 + \mu_4) > 0$ and $a_0 = \mu_1\mu_2\mu_3\mu_4 > 0$. By (1) the equation $F(a, b, h) = 0$ has at least two solutions in $h > 0$, so two μ_i are distinct positive numbers. It follows that the other two have negative real part, so we have (2). (4) and (5) follow easily.

To prove (6), we calculate that $\frac{\partial F}{\partial h} = \frac{a}{h^4}(h^4 + b_3h^3 + b_2h^2 + b_1h + b_0)$ with $b_3 = 0$, $b_2 > 0$, and $b_0 < 0$. Then $\frac{\partial F}{\partial h} = 0$ if and only if

$$h^4 + b_2h^2 + b_0 = -b_1h.$$

The graph of $y = h^4 + b_2h^2 + b_0$ is convex with a minimum at $(0, b_0)$, $b_0 < 0$. It follows easily that it meets the graph of $y = -b_1h$ in two points, one with $h > 0$, one with $h < 0$. ☐

5.3. **Connection Theorem.** Let $I = (h_-, h_+)$ be an open interval, with $h_- = -\infty$ and $h_+ = \infty$ allowed. Let $F : I \to \mathbb{R}$ be a C^1 function that satisfies the following conditions.

(C1) The equation $F(h) = 0$ has exactly two solutions in I. We denote them h_a and h_b, with $h_b < h_a$.
(C2) $F'(h_b) < 0$ and $F'(h_a) > 0$.
(C3) $\int_{h_-}^{h_a} F(h)dh > 0$ and $\int_{h_b}^{h_+} F(h)dh > 0$.

See Figure 5.1(a). For example, the third assumption is satisfied if the integrals are ∞.

Consider the system

$$\dot{h} = u, \tag{5.6}$$

$$\dot{u} = v, \tag{5.7}$$

$$\dot{v} = Bu - F(h), \tag{5.8}$$

with $B \ge 0$. This system has exactly two equilibria, at $(h_a, 0, 0)$ and $(h_b, 0, 0)$. One easily checks that both are hyperbolic, $W^u(h_a, 0, 0)$ has dimension two, and $W^s(h_b, 0, 0)$ has dimension two.

Theorem 5.2 (Connection Theorem)**.** *Assume (C1)–(C3). Then* $W^u(h_a,0,0)$ *and* $W^s(h_b,0,0)$ *have nonempty intersection.*

The system (5.6)–(5.8) is equivalent to the third-order equation $\dddot{h} - B\dot{h} + F(h) = 0$. There is a literature on connecting orbits for this equation that is larger than one might expect. For $B=0$ and $F(h)=h^2-1$, existence of a connecting orbit was proved by Kopell and Howard [3] using shooting, and later by Conley [2] using the Conley Index. Troy [18] gave a shooting argument for existence for $B=0$ and a different F. For $B=0$ and general F, Mock [14] gave a nice shooting argument, but his proof that a certain function is continuous (top of p. 387) is not correct. Michelson [12] gave a Conley index proof completely different from that in [2] for $B=0$ and general F; in fact, he allows a more general differential operator of odd order at least 3 in place of $\dddot{h}$. Renardy [15], motivated by a problem involving surfactants, treated a closely related situation. His proof would apply to $B \geq 0$ and general F. However, an inequality at the top of p. 291 cannot be used on an infinite interval.

We follow Michelson's argument, which easily generalizes to the case $B>0$. The proof uses a Lyapunov function and the Conley Index [2], and is given in nine steps. In step 1 we give the Lyapunov function. In steps 2–5 we find bounds on the set of bounded solutions of (5.6)–(5.8). In steps 6–9 we deform the set of bounded solutions to the empty set by perturbing (5.6)–(5.8), and use the Conley Index conclude the result.

Proof. 1. Let $G(h) = -\int F(h)\,dh$, and let $L(h,u,v) = -uv + G(h)$. See Figure 5.1(b). Then $\dot{L} = -(Bu^2+v^2)$, so L is decreasing along solutions of (5.6)–(5.8).

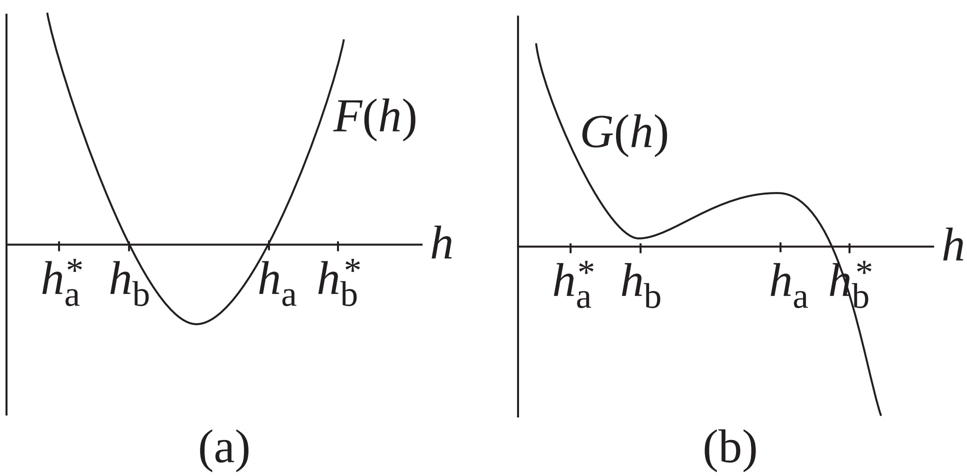


FIGURE 5.1. The functions $F(h)$ and $G(h)$ with $h_- = 0$ and $h_+ = \infty$.

2. Let $(h(t),u(t),v(t))$ be a bounded solution of (5.6)–(5.8). Then its α- and ω-limit sets are invariant sets contained in $\{(h,u,v) : \dot{L} = 0\}$. It follows easily that each is an equilibrium, so $\lim_{t\to\infty}(h(t),u(t),v(t))$ and $\lim_{t\to-\infty}(h(t),u(t),v(t))$ are both equilibria.

3. Using assumption (C3) we see that there exist h_a^*, $h_- < h_a^* < h_b$, such that $G(h_a^*) > G(h_a)$, and h_b^*, $h_a < h_b^* < h_+$, such that $G(h_b) > G(h_b^*)$. See Figure 5.1(b).

4. Let $(h(t),u(t),v(t))$ be a bounded solution of (5.6)–(5.8). We claim that $h_a^* < h(t) < h_b^*$ for all t.

The proof is by contradiction. Suppose $\max h(t) = h(t_1) \geq h_b^*$. Then $L(h(t_1),u(t_1),v(t_1)) = G(h(t_1))$ because $u(t_1) = \dot{h}(t_1) = 0$. Since $h(t_1) \geq h_b^*$, $G(h(t_1)) \leq G(h_b^*) < G(h_b)$, and of course $G(h_b) < G(h_a)$. But $L(h_b,0,0) = G(h_b)$ and $L(h_a,0,0) = G(h_a)$. Since L is decreasing on solutions, $(h(t),u(t),v(t))$ cannot approach an equilibrium as $t \to \infty$. This is a contradiction.

Similarly, if $\min h(t) \le h_a^*$, then $(h(t),u(t),v(t))$ cannot approach an equilibrium as $t \to -\infty$.

5. By step 4, for any bounded solution $(h(t),u(t),v(t))$ of (5.6)–(5.8), $\|h\|_\infty < h_b^*$. We claim that if M_0 is chosen sufficiently large, we have $\|\dot{h}\|_\infty = \|u\|_\infty < M_0$ and $\|\ddot{h}\|_\infty = \|v\|_\infty < M_0$.

The proof relies on Nirenberg's inequality in $\mathbb{R}$, which states that if

$$\frac{1}{p} = j + a\left(\frac{1}{r} - m\right) + (1-a)\frac{1}{q} \quad \text{and} \quad \frac{j}{m} \le a \le 1, \tag{5.9}$$

then there is a constant C such that

$$\|w^{(j)}\|_p \le C\|w^{(m)}\|_r^a \|w\|_q^{1-a} \tag{5.10}$$

whenever both sides of the inequality make sense [13].

Let $(h(t),u(t),v(t))$ be a bounded solution of (5.6)–(5.8). From steps 1 and 2,

$$\int_{-\infty}^{\infty} B(\dot{h}(t))^2 + (\ddot{h}(t))^2\,dt = -(L(\infty) - L(-\infty)) \le G(h_a) - G(h_b).$$

Therefore there is a uniform bound on $\|\ddot{h}\|_2$. By Nirenberg's inequality with $j=1$, $p=\infty$, $r=2$, $m=2$, $q=\infty$, and $a=\frac{2}{3}$,

$$\|\dot{h}\|_\infty \le C\|\ddot{h}\|_2^{\frac{2}{3}}\|h\|_\infty^{\frac{1}{3}}.$$

Therefore there is a uniform bound on $\|\dot{h}\|_\infty$.

Let $w = v = \ddot{h}$. By Nirenberg's inequality with $j=0$, $p=\infty$, $r=\infty$, $m=1$, $q=2$, and $a=\frac{1}{3}$,

$$\|\ddot{h}\|_\infty = \|w\|_\infty \le C\|\dot{w}\|_\infty^{\frac{1}{3}}\|w\|_2^{\frac{2}{3}} = C\|\dddot{h}\|_\infty^{\frac{1}{3}}\|\ddot{h}\|_2^{\frac{2}{3}}$$

Now the formula

$$\dddot{h} = B\dot{h} - F(h),$$

together with our uniform bounds on $\|h\|_\infty$ and $\|\dot{h}\|_\infty$, yields a uniform bound on $\|\dddot{h}\|_\infty$. This together with our uniform bound on $\|\ddot{h}\|_2$ yields a uniform bound on $\|\ddot{h}\|_\infty$.

6. Consider the one-parameter family of differential equations

$$\dot{h} = u, \tag{5.11}$$

$$\dot{u} = v, \tag{5.12}$$

$$\dot{v} = Bu - (F(h) + \mu), \tag{5.13}$$

Let $\mu_* = \min F(h)$. For $0 \le \mu \le \mu_*$, this system has one or more equilibria on the h-axis between $(h_b,0,0)$ and $(h_a,0,0)$, and no other equilibria; for $\mu > \mu_*$, it has no equilibria.

7. For each μ with $0 \le \mu \le \mu_*$, let E_μ denote the set of points (u,v,w) that lie in bounded solutions of (5.6)–(5.8).

From steps 4 and 5, E_0 lies in the interior of the closure of the open set $U_0 = \{(h,u,v) : h_a^* < h(t) < h_b^*,\ |u| < M_0,\ |v| < M_0\}$. E_0 is the largest invariant set contained in U_0, and U_0 is an isolating neighborhood of E_0.

By the same argument with minor modifications, for each μ with $0 \le \mu \le \mu_*$, there is a number $M_\mu > 0$ such that E_μ lies in the interior of the closure of an open set U_μ defined like U_0 except that M_0 is replaced by M_μ.

8. It is easy to see that M_μ can be chosen independently of μ for $0 \le \mu \le \mu_*$. Thus we can find a single open set U that serves as an isolating neighborhood for all the E_μ, $0 \le \mu \le \mu_*$.

9. For $\mu > \mu_*$, the system has no bounded solutions. It follows that the Conley index of E_0 equals the Conley index of the empty set. Since the Conley index of two hyperbolic fixed points does not equal the Conley index of the empty set, E_0 must contain a nontrivial solution connecting two equilibria. Because of the Lyapunov function L, it must connect $(h_a, 0, 0)$ to $(h_b, 0, 0)$.

□

5.4. **Transversality Theorem.** In addition to the hypotheses of the Connection Theorem, assume:

(C4) $B > 0$.
(C5) F has exactly one critical point h_c.

Of course, $h_b < h_c < h_a$.

If (C5) holds, we will call a solution of (5.6)–(5.8) that lies in $W^u(h_a, 0, 0) \cap W^s(h_b, 0, 0)$ a *simple connection* from $(h_a, 0, 0)$ to $(h_b, 0, 0)$ if there is exactly one time $t = t_c$ at which $h^*(t) = h_c$.

Theorem 5.3 (Transversality Theorem)**.** *Assume (C1)–(C5). Let $(h^*(t), u^*(t), v^*(t))$ be a simple connection from $(h_a, 0, 0)$ to $(h_b, 0, 0)$. Then $W^u(h_a, 0, 0)$ and $W^s(h_b, 0, 0)$ meet transversally along $(h^*(t), u^*(t), v^*(t))$.*

We remark that Kopell and Howard [3] prove a transversality result for the case $B = 0$ and $F(h) = h^2 - 1$, but their method does not appear to generalize.

Proof. Linearizing (5.6)–(5.8) along $(h^*(t), u^*(t), v^*(t))$, we obtain

$$\dot{\bar{h}} = \bar{u}, \quad \dot{\bar{u}} = \bar{v}, \quad \dot{\bar{v}} = B\bar{u} - F'(h^*(t))\bar{h}. \tag{5.14}$$

Note that $h^*(t)$ approaches h_a (respectively h_b) as $t \to -\infty$ (respectively $t \to \infty$). Since (5.14) with $h^*(t)$ replaced by h_a (respectively h_b has two eigenvalues with positive (respectively negative) real part, this linear system has a two-dimensional space of solutions that approach 0 exponentially as $t \to -\infty$, and a two-dimensional space of solutions that approach 0 exponentially as $t \to \infty$. We must show: (I) these two-dimensional spaces meet transversally.

Given a real linear differential equation $\dot{x} = L(t)x$, the adjoint system is $\dot{\psi} = -\psi L(t)$ The adjoint system for (5.14) is

$$\dot{\psi}_1 = F'(h^*(t))\psi_3, \quad \dot{\psi}_2 = -\psi_1 - B\psi_3, \quad \dot{\psi}_3 = -\psi_2. \tag{5.15}$$

This linear system has a one-dimensional space of solutions that approach 0 exponentially as $t \to -\infty$, and a one-dimensional space of solutions that approach 0 exponentially as $t \to \infty$. Equivalently to (I), we must show: (II) these one-dimensional spaces have trivial intersection.

The equation (5.15), like any nonautonomous linear differential equation, defines a nonautonomous differential equation $\dot{p} = f(t, p)$ on $\mathbb{RP}^2$, the space of lines through the origin in $\mathbb{R}^3$. A point of $\mathbb{RP}^2$ represents an equivalence class of vectors in $\mathbb{R}^3 \setminus \{0\}$; we write $p = [\psi]$, with $\psi \sim \tilde{\psi}$ is there is a number $s \neq 0$ such that $\psi = s\tilde{\psi}$. Let p_a (respectively p_b) denote the one-dimensional eigenspace for the unique positive (respectively negative) eigenvalue of (5.15) at $t = -\infty$ (respectively $t = \infty$), i.e., the positive (respectively negative) eigenvalue of (5.15) when $h^*(t)$ is replaced by h_a (respectively h_b). Equivalently to (II), we must show: (III) no solution of $\dot{p} = f(t,p)$ approaches p_a as $t \to -\infty$ and p_b as $t \to \infty$.

$\mathbb{RP}^2$ is covered by three coordinates systems; these coordinates, and the differential equation $\dot{p} = f(t,p)$ in each, are:

(1) $\phi = (\phi_1, \phi_2)$, $\phi_1 = \frac{\psi_1}{\psi_3}$, $\phi_2 = \frac{\psi_2}{\psi_3}$; $\dot{\phi}_1 = F'(h^*(t)) + \phi_1\phi_2$, $\dot{\phi}_2 = -\phi_1 - B + \phi_2^2$.
(2) $\eta = (\eta_1, \eta_3)$, $\eta_1 = \frac{\psi_1}{\psi_2}$, $\eta_3 = \frac{\psi_3}{\psi_2}$; $\dot{\eta}_1 = F'(h^*(t))\eta_3 + \eta_1^2 + B\eta_1\eta_3$, $\dot{\eta}_3 = -1 + \eta_1\eta_3 + B\eta_3^2$.
(3) $\chi = (\chi_2, \chi_3)$, $\chi_2 = \frac{\psi_2}{\psi_1}$, $\chi_3 = \frac{\psi_3}{\psi_1}$, $\dot{\chi}_2 = -1 - B\chi_3 - F'(h^*(t))\chi_2\chi_3$, $\dot{\chi}_3 = -\chi_2 - F'(h^*(t))\chi_3^2$.

Let $U_a = \{p \in \mathbb{RP}^2 : p = [\psi] \text{ with } \psi_1 > 0,\ \psi_2 < 0, \text{ and } \psi_3 > 0\}$, and let $U_b = \{p \in \mathbb{RP}^2 : p = [\psi] \text{ with } \psi_1 > 0,\ \psi_2 > 0, \text{ and } \psi_3 > 0\}$. U_a and U_b are disjoint and open in $\mathbb{RP}^2$, and it is straightforward to check that $p_a \in U_a$ and $p_b \in U_b$. More precisely, consider the system of equations

$$F'(h) + \phi_1\phi_2 = 0, \quad -\phi_1 - B + \phi_2^2 = 0, \tag{5.16}$$

derived by setting the right hand side of the system in ϕ-coordinates equal to 0. For each $h \neq h_c$, there is a unique solution $\phi(h)$ of (5.16) with $\phi_1 > 0$. (The system has 0, 1, or 2 solutions with $\phi_1 < 0$.) Then $\phi_2(h)$ has sign opposite that of $F'(h)$. We have $p_a = [(\phi_1(h_a), \phi_2(h_a), 1)]$ and $p_b = [(\phi_1(h_b), \phi_2(h_b), 1)]$.

We claim that (A) there is no solution of $\dot{p} = f(t,p)$ that lies in U_a for large negative t and in U_b for large positive t. This implies (III).

To prove (A), we will show the the following. Suppose $p(t)$ is a solution of $\dot{p} = f(t,p)$ that lies in U_a for large negative t, and $q(t)$ is a solution of $\dot{p} = f(t,p)$ that lies in U_b for large positive t. Then (a) $p(t) \in U_a$ for $t < t_c$, (b) $q(t) \in U_b$ for $t > t_c$, and (c) $p(t_c) \neq q(t_c)$.

Let us first show (a). Suppose $p(t)$ is a solution of $\dot{p} = f(t,p)$ that lies in U_a for large negative t. Suppose $p(t)$ first reaches the boundary of U_a at time $t < t_c$. Since $t < t_c$, assumption (4) implies that $h^*(t) > h_c$, so $F'(h^*(t)) > 0$. We consider each coordinate patch. In ϕ-coordinates, U_a corresponds to $\{(\phi_1, \phi_2) : \phi_1 > 0 \text{ and } \phi_2 < 0\}$. On the boundary of this set:

- If $t < t_c$, $\phi_1 \geq 0$ and $\phi_2 = 0$, then $\dot{\phi}_2 < 0$.
- If $t < t_c$, $\phi_1 = 0$ and $\phi_2 < 0$, then $\dot{\phi}_1 > 0$.

Therefore, $p(t)$ does not leave U_a through its boundary in the ϕ-coordinate patch at a time $t < t_c$. A similar argument shows that $p(t)$ does not leave U_a through its boundary in the other coordinate patches at a time $t < t_c$. This shows (a).

A similar argument shows (b), i.e, we show that $q(t)$ does not leave U_b through its boundary in backward time at any time $t > t_c$.

It remains to show (c). Suppose $p(t)$ first leaves U_a at time $t = t_c$ and $q(t)$ first leaves U_b in backward time at time $t = t_c$. Suppose that $p(t_c)$ equals $q(t_c)$. We consider the different coordinate patches. Suppose $p(t_c) = q(t_c)$ is a point of the ϕ-coordinate patch. In ϕ-coordinates, U_a corresponds to $\{(\phi_1, \phi_2) : \phi_1 > 0 \text{ and } \phi_2 < 0\}$, and U_b corresponds to $\{(\phi_1, \phi_2) : \phi_1 > 0 \text{ and } \phi_2 > 0\}$. Their common boundary is the ray $\{(\phi_1, \phi_2) : \phi_1 \geq 0 \text{ and } \phi_2 = 0\}$. On this ray, for $t = t_c$, $\dot{\phi}_2 < 0$. (At the origin we need $B > 0$ at this point.)

But then neither $p(t_c)$ nor $q(t_c)$ can belong to this ray, so they are certainly not equal. A similar argument applies to the other coordinate patches. □

6. Invariant spaces and equilibria

For the system (5.2)–(5.5), the three-dimensional spaces $\gamma = 0$ (no surfactant) and $\gamma = 1$ (infinite surfactant) are invariant.

For every $\tilde{C} \geq 0$, $\tilde{D} \geq 0$, and $\tilde{\beta} \geq 0$, there are equilibria of (5.2)–(5.5) at $(h, u, v, \gamma) = (h_L, 0, 0, 0)$, $(h_R, 0, 0, 0)$, $(h_1, 0, 0, 1)$, and $(h_2, 0, 0, 1)$. If $\tilde{D} > 0$, these are the only equilibria. Traveling waves (2.8) of (2.2)–(2.3) correspond to solutions of (5.2)–(5.5) that go from $(h_L, 0, 0, 0)$ to $(h_R, 0, 0, 0)$.

For $\tilde{C} > 0$, the linearization of (5.2)–(5.5) at $(h_L, 0, 0, 0)$ or $(h_R, 0, 0, 0)$ has the matrix

$$\begin{pmatrix} 0 & 4\tilde{D} & 0 & 0 \\ 0 & 0 & 4\tilde{D} & 0 \\ -4\tilde{C}^{-1}\tilde{D}H'(h) & 4\tilde{C}^{-1}\tilde{D}\tilde{\beta} & 0 & -\tilde{C}^{-1}hP(h) \\ 0 & 0 & 0 & Q(h) \end{pmatrix}. \tag{6.1}$$

Characteristic equation:

$$(\lambda - Q(h))(\lambda^3 - 16\tilde{C}^{-1}\tilde{D}^2\tilde{\beta}\lambda + 64\tilde{C}^{-1}\tilde{D}^3H'(h)) = 0. \tag{6.2}$$

Let $p(\lambda) = \lambda^3 - 16\tilde{C}^{-1}\tilde{D}^2\tilde{\beta}\lambda + 64\tilde{C}^{-1}\tilde{D}^3H'(h)$ have roots $\lambda_1, \lambda_2, \lambda_3$. Then $\lambda_1 + \lambda_2 + \lambda_3 = 0$, and, for $\tilde{C} > 0$ and $\tilde{D} > 0$, the sign of $\lambda_1\lambda_2\lambda_3$ is opposite that of $H'(h)$.

Conclusions for $\tilde{C} > 0$ and $\tilde{D} > 0$:

(1) At $(h_L, 0, 0, 0)$ there are three eigenvalues with positive real part and one with negative real part.
(2) At $(h_R, 0, 0, 0)$ there are one eigenvalue with positive real part and three with negative real part.

Thus for $\tilde{C} > 0$ and $\tilde{D} > 0$, $W^u(h_L, 0, 0, 0)$ and $W^s(h_R, 0, 0, 0)$ are three-dimensional. If they are transverse, the intersection is two-dimensional. We also note:

(1) At $(h_L, 0, 0, 0)$, two of the eigenvalues with positive real part are complex if $4\tilde{\beta}^3 < 27\tilde{C}H'(h_L)^2$.
(2) At $(h_L, 0, 0, 0)$, two of the eigenvalues with negative real part are complex if $4\tilde{\beta}^3 < 27\tilde{C}H'(h_R)^2$.

In Region 1, for fixed h_L (respectively h_R), complex eigenvalues occur for $\tilde{\beta}_1$ less than a value of order one. In Region 3, for fixed h_L (respectively h_R), complex eigenvalues occur for $\tilde{C}_2$ greater than a number of order one times $\tilde{\beta}_2^3$. In the other regions, for fixed h_L (respectively h_R), complex eigenvalues do not occur provided the constants in the definition of the region are taken sufficiently small.

7. Connecting orbits with Γ small

We consider the traveling wave system (3.11)–(3.14) with the normalization (N1). Hence $\tilde{C} = 1$, $\tilde{\beta} = \tilde{\beta}_1 \geq 0$, and we assume $\tilde{D} = \tilde{D}_1 > 0$:

$$\dot{h} = (h\Gamma + 4\tilde{D}_1)u, \tag{7.1}$$

$$\dot{u} = (h\Gamma + 4\tilde{D}_1)v, \tag{7.2}$$

$$\dot{v} = h\Gamma\left(\tilde{\beta}_1 u - P(h)\right) + 4\tilde{D}_1\left(\tilde{\beta}_1 u - H(h)\right), \tag{7.3}$$

$$\dot{\Gamma} = \Gamma Q(h). \tag{7.4}$$

The space $\Gamma = 0$ is invariant. Restricting (7.1)–(7.4) to $\Gamma = 0$ and dividing by $4\tilde{D}_1 > 0$, we obtain

$$\dot{h} = u, \tag{7.5}$$

$$\dot{u} = v, \tag{7.6}$$

$$\dot{v} = \tilde{\beta}_1 u - H(h). \tag{7.7}$$

The only equilibria of (7.5)–(7.7) are $(h_L, 0, 0)$ and $(h_R, 0, 0)$. Both are hyperbolic. The former has two-dimensional unstable manifold; the latter has two-dimensional stable manifold. By the Connection Theorem, $W^u(h_L, 0, 0) \cap W^s(h_R, 0, 0)$ is nonempty.

Let C be one of the curves in the intersection. Suppose $W^u(h_L, 0, 0)$ and $W^s(h_R, 0, 0)$ meet transversally along C. (According to the Transversality Theorem, this is the case if $\tilde{\beta}_1 > 0$ and C is a simple connection from $(h_L, 0, 0)$ to $(h_R, 0, 0)$.) Then for the system (7.1)–(7.4), the three-dimensional manifolds $W^u(h_L, 0, 0, 0)$ and $W^s(h_R, 0, 0, 0)$ meet transversally in a two-dimensional manifold of connecting orbits that includes C. Thus (3.11)–(3.14) has a one-parameter family of connecting orbits from $(h_L, 0, 0, 0)$ to $(h_R, 0, 0, 0)$.

Note that this argument proves the existence only of connecting orbits with small Γ. Moreover, if $\tilde{D}_1$ approaches 0, some eigenvalues at the equilibria, which satisfy (6.2), approach 0. Therefore the size of the Γ-interval for which connections are guaranteed to exist by the argument of this section goes to 0.

8. Region 1

We consider the normalized traveling wave system (7.1)–(7.4) of the previous section. In Region 1, $\tilde{\beta}_1$ lies in a given bounded interval $0 < \tilde{\beta}_1 \leq \eta^{-1}$, and $\tilde{D}_1$ is small.

For $\tilde{D}_1 = 0$, (7.1)–(7.4) becomes

$$\dot{h} = h\Gamma u, \tag{8.1}$$

$$\dot{u} = h\Gamma v, \tag{8.2}$$

$$\dot{v} = h\Gamma\left(\tilde{\beta}_1 u - P(h)\right), \tag{8.3}$$

$$\dot{\Gamma} = \Gamma Q(h). \tag{8.4}$$

The space $\Gamma = 0$ consists of equilibria of (8.1)–(8.4). It is normally hyperbolic for h away from h_*. Thus for small $\tilde{D}_1$, the system (7.1)–(7.4), although not a slow-fast system, has the essential structure of one: for $\tilde{D}_1 = 0$ it has a large manifold of equilibria.

In (7.1)–(7.4), $\Gamma = 0$ remains invariant for $\tilde{D}_1 > 0$. Restricting (7.1)–(7.4) to $\Gamma = 0$ and dividing by $4\tilde{D}_1$, we obtain (7.5)–(7.7). This system plays the role of the slow system. The "fast system" is (8.1)–(8.4).

Let $W^u_{\text{loc}}(h_L, 0, 0)$ denote an open subset of the two-dimensional unstable manifold of $(h_L, 0, 0)$ whose closure is contained in $\{(h, u, v) : h > h_*\}$. Similarly, let $W^s_{\text{loc}}(h_R, 0, 0)$ denote an open subset of the two-dimensional stable manifold of $(h_R, 0, 0)$ whose closure is contained in $\{(h, u, v) : h < h_*\}$. For $\tilde{D}_1 = 0$ (i.e., for (8.1)–(8.4)), $W^u_{\text{loc}}(h_L, 0, 0) \times \{0\}$ (respectively $W^s_{\text{loc}}(h_R, 0, 0) \times \{0\}$) is a two-dimensional manifold of equilibria that has a three-dimensional unstable (respectively stable) manifold in $huv\Gamma$-space.

For $\tilde{D}_1 > 0$ the manifolds $W^u_{\text{loc}}(h_L, 0, 0) \times \{0\}$ and $W^s_{\text{loc}}(h_R, 0, 0) \times \{0\}$ remain invariant because the system (7.1)–(7.4), after restriction to $\Gamma = 0$ and division by $\tilde{D}_1$, is (7.5)–(7.7) independent of $\tilde{D}_1$. The unstable manifold of $W^u_{\text{loc}}(h_L, 0, 0) \times \{0\}$ and the stable manifold of $W^s_{\text{loc}}(h_R, 0, 0) \times \{0\}$ respectively perturb slightly. Therefore, if these three-dimensional manifolds intersect transversally for $\tilde{D}_1 = 0$ (i.e., for (8.1)–(8.4)), then they do so for small $\tilde{D}_1 > 0$, and we have a two-dimensional manifold of connecting orbits from $(h_L, 0, 0, 0)$ to $(h_R, 0, 0, 0)$. In this case connecting orbits for $\tilde{D}_1 > 0$ are close to the following singular connecting orbits for $\tilde{D}_1 = 0$: : (1) a solution of the slow system (7.5)–(7.7) from $(h_L, 0, 0)$ to a point (h_0, u_0, v_0) in $W^u_{\text{loc}}(h_L, 0, 0)$, the unstable manifold of $(h_L, 0, 0)$ for the slow system; (2) a connecting orbit of the fast system (8.1)–(8.4) from $(h_0, u_0, v_0, 0)$ to $(h_1, u_1, v_1, 0)$, where $(h_1, u_1, v_1) \in W^s_{\text{loc}}(h_R, 0, 0)$, the stable manifold of $(h_R, 0, 0)$ for the slow system; (3) a solution of the slow system from (h_1, u_1, v_1) to $(h_R, 0, 0)$.

One way to study connecting orbits of (8.1)–(8.4) from $\Gamma = 0$ to itself is to first divide (8.1)–(8.4) by $h\Gamma$:

$$\dot{h} = u, \tag{8.5}$$

$$\dot{u} = v, \tag{8.6}$$

$$\dot{v} = \tilde{\beta}_1 u - P(h), \tag{8.7}$$

$$\dot{\Gamma} = \frac{1}{h} Q(h). \tag{8.8}$$

The system (8.5)–(8.8) is solved with initial condition $(h, u, v, \Gamma)(0) = (h_0, u_0, v_0, 0)$ and $h_0 > h_*$. (Actually, the system for (h, u, v) is independent of Γ.) Suppose there is a $t_1 > 0$ such that $\Gamma(t_1) = 0$. Let $(h_1, u_1, v_1) = (h, u, v)(t_1)$. Then there is a connecting orbit of (8.1)–(8.4) from $(h_0, u_0, v_0, 0)$ to $(h_1, u_1, v_1, 0)$. We define $\pi(h_0, u_0, v_0) = (h_1, u_1, v_1)$. If the two-dimensional manifold $\pi(W^u_{\text{loc}}(h_L, 0, 0))$ is transverse to the two-dimensional manifold $W^s_{\text{loc}}(h_R, 0, 0))$ in huv-space, then the desired transversality condition holds. See Figure 8.1. However, we do not see how to show that this is the case.

9. Region 2

Theorem 9.1. *Let h_L and h_R satisfying* (3.15) *be given. Consider the traveling wave system* (5.2)–(5.5) *with the normalization (N3): $\tilde{C} = \tilde{C}_3 \geq 0$, $\tilde{D} = \tilde{D}_3 \geq 0$, and $\tilde{\beta} = 1$. Then for each η_2 with $0 < \eta_2 < 1$, there exists $\delta_2 > 0$ such that, if Region 2 is defined using these values, then for each parameter triple $(\tilde{C}_3, \tilde{D}_3, 1)$ in Region 2, there exists a one-parameter family of traveling waves connecting h_L to h_R. Moreover,* $\max \gamma$ *ranges from 0 to 1.*

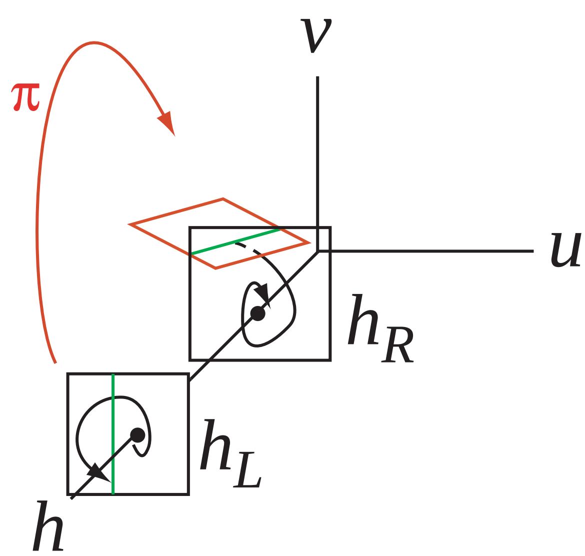


Figure 8.1

Proof. In the traveling wave system (5.2)–(5.5) , let

$$c = \tilde{C}^{\frac{1}{2}}, \quad \hat{v} = cv.$$

After dropping the hat and multiplying by c, we obtain the system

$$\dot{h} = c(h\gamma + 4\tilde{D}(1-\gamma))u, \tag{9.1}$$

$$\dot{u} = (h\gamma + 4\tilde{D}(1-\gamma))v, \tag{9.2}$$

$$\dot{v} = h\gamma\left(\tilde{\beta}u - P(h)\right) + 4\tilde{D}(1-\gamma)\left(\tilde{\beta}u - H(h)\right), \tag{9.3}$$

$$\dot{\gamma} = c\gamma(1-\gamma)^2 Q(h). \tag{9.4}$$

Using the normalized parameters (N3), (9.1)–(9.4) becomes

$$\dot{h} = c(h\gamma + 4\tilde{D}_3(1-\gamma))u, \tag{9.5}$$

$$\dot{u} = (h\gamma + 4\tilde{D}_3(1-\gamma))v, \tag{9.6}$$

$$\dot{v} = h\gamma\left(u - P(h)\right) + 4\tilde{D}_3(1-\gamma)\left(u - H(h)\right), \tag{9.7}$$

$$\dot{\gamma} = c\gamma(1-\gamma)^2 Q(h), \tag{9.8}$$

with $c = \tilde{C}_3^{\frac{1}{2}}$. To study Region 2, given η_2 with $0 < \eta_2 < 1$, we assume $\eta_2 \leq \tilde{D}_3 \leq \eta_2^{-1}$ and c is small. Since c is small, we have a slow-fast system with two slow variables (h and γ) and two fast variables (u and v).

Set $c = 0$ in (9.5)–(9.8):

$$\dot{h} = 0, \tag{9.9}$$
$$\dot{u} = (h\gamma + 4\tilde{D}_3(1-\gamma))v, \tag{9.10}$$
$$\dot{v} = h\gamma\,(u - P(h)) + 4\tilde{D}_3(1-\gamma)\,(u - H(h))\,, \tag{9.11}$$
$$\dot{\gamma} = 0. \tag{9.12}$$

The set

$$u = \frac{h\gamma P(h) + 4\tilde{D}_3(1-\gamma)H(h)}{h\gamma + 4\tilde{D}_3(1-\gamma)}, \quad v = 0, \quad h > 0, \quad 0 \le \gamma \le 1 \tag{9.13}$$

is a normally hyperbolic manifold of equilibria of dimension 2. (One positive eigenvalue, one negative eigenvalue.)

Perturbed normally hyperbolic invariant manifold for small $c > 0$:

$$u = K(h,\gamma,c,\tilde{D}_3), \quad K(h,\Gamma,0,\tilde{D}_3) = \frac{h\gamma P(h) + 4\tilde{D}_3(1-\gamma)H(h)}{h\gamma + 4\tilde{D}_3(1-\gamma)}, \tag{9.14}$$
$$v = L(h,\gamma,c,\tilde{D}_3), \quad L(h,\gamma,0,\tilde{D}_3) = 0. \tag{9.15}$$

Since $(h,u,v,\gamma) = (h_L,0,0,0)$ and $(h_R,0,0,0)$ are equilibria for all $(c,\tilde{D}_3)$,

$$K(h_L,0,c,\tilde{D}_3) = L(h_L,0,c,\tilde{D}_3) = K(h_R,0,c,\tilde{D}_3) = L(h_R,0,c,\tilde{D}_3) = 0.$$

Since $(h,u,v,\gamma) = (h_1,0,0,1)$ and $(h_2,0,0,1)$ are equilibria for all $(c,\tilde{D}_3)$,

$$K(h_1,1,c,\tilde{D}_3) = L(h_1,1,c,\tilde{D}_3) = K(h_2,1,c,\tilde{D}_3) = L(h_2,1,c,\tilde{D}_3) = 0.$$

The vector field on the perturbed normally hyperbolic invariant manifold is

$$\begin{aligned}\dot{h} &= c(h\gamma + 4\tilde{D}_3(1-\gamma))K(h,\Gamma,c,\tilde{D}_3)\\ &= c\big(h\gamma P(h) + 4\tilde{D}_3(1-\gamma)H(h) + O(c)\big),\end{aligned} \tag{9.16}$$
$$\dot{\gamma} = c\gamma(1-\gamma)^2 Q(h). \tag{9.17}$$

In slow time (i.e., divide by c):

$$h' = h\gamma P(h) + 4\tilde{D}_3(1-\gamma)H(h) + O(c), \tag{9.18}$$
$$\gamma' = \gamma(1-\gamma)^2 Q(h). \tag{9.19}$$

For all $(c,\tilde{D}_3)$ there are equilibria of (9.18)–(9.19) at $(h,\Gamma) = (h_L,0)$, $(h_R,0)$, $(h_1,1)$, and $(h_2,1)$. The first is a repeller, the second is an attractor, and the last two are semihyperbolic.

For $c = 0$, the flow can be drawn with the help of Proposition 5.1. In the notation of that theorem, the nullclines are the curves $h = h_i(\gamma, 4\tilde{D}_3(1-\gamma))$, $i = 1$, 2, and $h = h_*$; $h_2(\gamma,4\tilde{D}_3(1-\gamma)) < h_* < h_1(\gamma,4\tilde{D}_3(1-\gamma))$. In particular:

- For $0 < h < h_2(\gamma,4\tilde{D}_3(1-\gamma))$, $\dot{h} > 0$ and $\dot{\gamma} < 0$.
- For $h_2(\gamma,4\tilde{D}_3(1-\gamma)) < h < h_*$, $\dot{h} < 0$ and $\dot{\gamma} < 0$.
- For $h_* < h < h_1(\gamma,4\tilde{D}_3(1-\gamma))$, $\dot{h} < 0$ and $\dot{\gamma} > 0$.
- For $h_1(\gamma,4\tilde{D}_3(1-\gamma)) < h$, $\dot{h} > 0$ and $\dot{\gamma} > 0$.

The center manifold of $(h_1, 1)$ (respectively $(h_2, 1)$), which is a separatrix, connects to $(h_L, 0)$ (respectively $(h_R, 0)$). The flow is shown in Figure 9.1: there is a one-parameter family of connections from $(h_L, 0)$ to $(h_R, 0)$ that fills the region $h_2(\gamma, 4\tilde{D}_3(1-\gamma)) < h < h_1(\gamma, 4\tilde{D}_3(1-\gamma))$, $0 < \gamma < 1$. The flow for c small is exactly the same. □

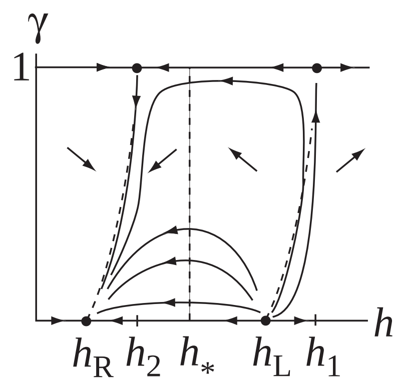


FIGURE 9.1. Flow of (9.18)–(9.19) for small $c \geq 0$. Nullclines are dashed. Center manifolds of the equilibria at $(h_1, 0)$ and $(h_2, 0)$ are shown.

For $c = 0$, the eigenvalues of the linearization of (9.18)–(9.19) at $(h_L, 0)$ and $(h_R, 0)$ are $4\tilde{D}_3 H'(h)$ and $Q(h)$. Both are positive at h_L; both are negative at h_R. Their relative size affects the shape of solution curves in Figure 9.1:

(1) If $0 < 4\tilde{D}_3 H'(h_L) < Q(h_L)$, all integral curves in $h > 0$ but one that approach $(h_L, 0)$ as $t \to -\infty$ do so tangent to the h-axis.
(2) If $0 < Q(h_L) < 4\tilde{D}_3 H'(h_L)$, all integral curves in $h > 0$ that approach $(h_L, 0)$ as $t \to -\infty$ do so tangent the eigendirection for the eigenvalue $Q(h_L)$, which is $(-h_L P(h_L), 4\tilde{D}_3 H'(h_L) - Q(h_L))$. Both components are positive.
(3) If $Q(h_R) < 4\tilde{D}_3 H'(h_R) < 0$, all solutions in $h > 0$ but one that approach $(h_R, 0)$ as $t \to \infty$ do so tangent to the h-axis.
(4) If $4\tilde{D}_3 H'(h_R) < Q(h_R) < 0$, all solutions in $h > 0$ that approach $(h_R, 0)$ as $t \to \infty$ do so tangent to the eigendirection for the eigenvalue $Q(h_R)$, which is $(h_R P(h_R), Q(h_R) - 4\tilde{D}_3 H'(h_R))$. Both components are positive.

The first and third cases, which occur for small $\tilde{D}_3$, give rise to "feet" in the solution. The second and fourth cases, which occur for larger $\tilde{D}_3$, do not. However, in the second and fourth cases, the eigenvector has positive slope; this produces, at the left, a rise in the traveling wave above h_L, and, at the right, a slow descent toward h_R.

10. REGION 3

In the traveling wave system (5.2)–(5.5), let

$$\epsilon = \left(\frac{\tilde{C}}{\tilde{\beta}^3}\right)^{\frac{1}{2}}, \quad \hat{v} = \epsilon v.$$

After dropping the hat and multiplying by ϵ, we obtain the system

$$\dot{h} = \epsilon(h\gamma + 4\tilde{D}(1-\gamma))u, \tag{10.1}$$
$$\dot{u} = (h\gamma + 4\tilde{D}(1-\gamma))v, \tag{10.2}$$
$$\dot{v} = h\gamma\,(u - P(h)) + 4\tilde{D}(1-\gamma)\,(u - H(h))\,, \tag{10.3}$$
$$\dot{\gamma} = \tilde{\beta}\epsilon\gamma(1-\gamma)^2 Q(h). \tag{10.4}$$

We use the normalized parameters (N2): $\tilde{C} = \tilde{C}_2 \geq 0$, $\tilde{D} = 1$, and $\tilde{\beta} = \tilde{\beta}_2 \geq 0$. Thus (10.1)–(10.4) becomes

$$\dot{h} = \epsilon(h\gamma + 4(1-\gamma))u, \tag{10.5}$$
$$\dot{u} = (h\gamma + 4(1-\gamma))v, \tag{10.6}$$
$$\dot{v} = h\gamma\,(u - P(h)) + 4(1-\gamma)\,(u - H(h))\,, \tag{10.7}$$
$$\dot{\gamma} = \tilde{\beta}_2\epsilon\gamma(1-\gamma)^2 Q(h), \tag{10.8}$$

with $\epsilon = \left(\frac{\tilde{C}_2}{\tilde{\beta}_2^3}\right)^{\frac{1}{2}}$. To study Region 3, given η_2 with $0 < \eta_2 < 1$, we assume $\eta_2 \leq \frac{\tilde{C}_2}{\tilde{\beta}_2^3} \leq \eta_2^{-1}$ and $\tilde{\beta}_2$ is small. Therefore $\eta_2^{\frac{1}{2}} \leq \epsilon \leq \eta_2^{-\frac{1}{2}}$. Hence we have a slow-fast system with one slow variable (γ) and three fast variables (h, u, and v).

Set $\tilde{\beta}_2 = 0$:

$$\dot{h} = \epsilon(h\gamma + 4(1-\gamma))u, \tag{10.9}$$
$$\dot{u} = (h\gamma + 4(1-\gamma))v, \tag{10.10}$$
$$\dot{v} = h\gamma\,(u - P(h)) + 4(1-\gamma)\,(u - H(h))\,, \tag{10.11}$$
$$\dot{\gamma} = 0. \tag{10.12}$$

Each plane $\gamma =$ constant is invariant and has equilibria at points (h,u,v) with $h\gamma P(h) + 4(1-\gamma)H(h) = 0$ and $u = v = 0$. By Proposition 5.1, the equilibria constitute two one-dimensional normally hyperbolic invariant manifolds:

$$C_i = \{(h,u,v,\gamma) : h = h_i(\gamma, 4(1-\gamma)),\ u = 0,\ v = 0,\ 0 \leq \gamma \leq 1\}, \quad i = 1,\,2.$$

See Figure 10.1. Equilibria on C_1 have two eigenvalues with positive real part and one with negative real part; equilibria on C_2 have one eigenvalue with positive real part and two with negative real part. On the plane $\gamma =$ constant, after division by $h\gamma + 4(1-\gamma) > 0$ the system becomes

$$\dot{h} = \epsilon u, \tag{10.13}$$
$$\dot{u} = v, \tag{10.14}$$
$$\dot{v} = \frac{h\gamma}{h\gamma + 4(1-\gamma)}\,(u - P(h)) + \frac{4(1-\gamma)}{h\gamma + 4(1-\gamma)}\,(u - H(h))\,. \tag{10.15}$$

By a simple change of variables we can convert ϵ in (10.13) to 1. Then by the Connection Theorem, whose hypotheses are verified by Proposition 5.1, the two-dimensional unstable manifold of each equilibrium on C_1 meets the two-dimensional stable manifold of the equilibrium on C_2 with the same value of γ.

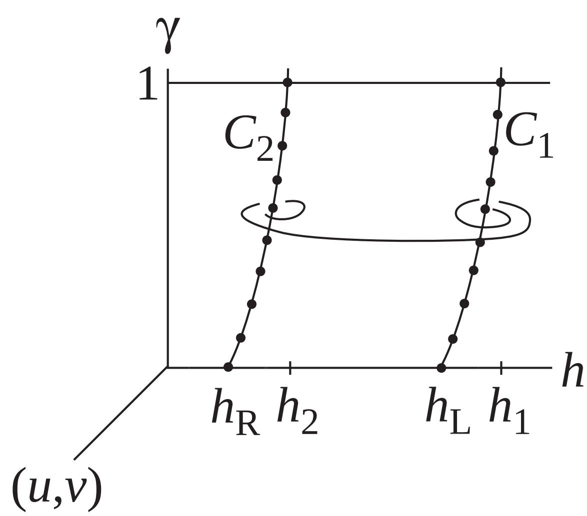


FIGURE 10.1

Theorem 10.1. *Let h_L and h_R satisfying* (3.15) *be given. Consider the traveling wave system* (5.2)–(5.5) *with the normalization (N2). Assume there is a γ_0, $0 \leq \gamma_0 \leq 1$, such that for* (10.13)–(10.15) *with $\gamma = \gamma_0$, the two-dimensional unstable manifold of the equilibrium on C_1 meets the two-dimensional stable manifold of the equilibrium on C_2 transversally. Then for each η_3 with $0 < \eta_3 < 1$, there exists $\delta_3 > 0$ such that, if Region 3 is defined using these values, then for each parameter triple $(\tilde{C}_2, 1, \tilde{\beta}_2)$ in Region 3, there exists a one-parameter family of traveling waves connecting h_L to h_R with* $\max \gamma$ *near γ_0.*

We remark that according to the Transversality Theorem, the transversality condition within the plane $\gamma = \gamma_0$ is satisfied if the connection there is simple.

Proof. The assumptions imply that for (10.13)–(10.15), the three-dimensional unstable manifold of C_1 meets the three-dimensional stable manifold of C_2 transversally near $\gamma = \gamma_0$ in a two-dimensional family of fast connecting orbits.

For small $\tilde{\beta}_2 > 0$, C_1 (respectively C_2) perturbs to a normally hyperbolic invariant curve connecting $(h_L, 0, 0)$ to $(h_1, 0, 0)$ (respectively $(h_R, 0, 0)$ to $(h_2, 0, 0)$). On the perturbed normally hyperbolic invariant manifolds corresponding to C_i, we have

$$\dot{\gamma} = \tilde{\beta}_2 \epsilon \gamma (1-\gamma)^2 Q(h_i(\gamma, 4(1-\gamma)) + O(\tilde{\beta}_2)),$$

which is positive on the interior of C_1 and negative on the interior of C_2. There is a repeller at h_L and an attractor at h_R.

Singular solutions ($\tilde{\beta}_2 = 0$) from $(h_L, 0, 0, 0)$ to $(h_R, 0, 0, 0)$ consist of a slow solution on C_1 from $(h_L, 0, 0, 0)$ to level γ, $0 < \gamma < 1$; a fast connection from C_1 to C_2 at level γ, and a slow solution on C_2 to $(h_R, 0, 0, 0)$. Because of the transversality, for γ near γ_0 they persist for small $\tilde{\beta}_2 > 0$. □

Note that at $(h_L, 0)$, C_1 has slope $\frac{-4H'(h_L)}{h_L P(h_L)} > 0$; at $(h_R, 0)$, C_2 has slope $\frac{-4H'(h_R)}{h_R P(h_R)} > 0$. As in the previous section, these slopes produce, at the left, a rise in the traveling wave above h_L, and, at the right, a slow descent toward h_R.

11. Region 4

Theorem 11.1. *Let h_L and h_R satisfying* (3.15) *be given. Consider the traveling wave system* (5.2)–(5.5) *with the normalization (N2): $\tilde{C} = \tilde{C}_2 \geq 0$, $\tilde{D} = 1$, and $\tilde{\beta} = \tilde{\beta}_2 \geq 0$. Then for each η_4 with $0 < \eta_4 < 1$, there exists $\delta_4 > 0$ such that, if Region 4 is defined using these values, then for each parameter triple $(\tilde{C}_2, 1, \tilde{\beta}_2)$ in Region 4, there exists a one-parameter family of traveling waves connecting h_L to h_R. Moreover,* $\max \gamma$ *ranges from 0 to 1.*

Proof. In the traveling wave system (10.5)–(10.8), with $\epsilon = \left(\frac{\tilde{C}_2}{\tilde{\beta}_2^3}\right)^{\frac{1}{2}}$, let

$$\beta_2 = \frac{\tilde{\beta}_2}{\epsilon} = \left(\frac{\tilde{\beta}_2^5}{\tilde{C}_2}\right)^{\frac{1}{2}}.$$

The system becomes

$$\dot{h} = \epsilon(h\gamma + 4(1-\gamma))u, \tag{11.1}$$
$$\dot{u} = (h\gamma + 4(1-\gamma))v, \tag{11.2}$$
$$\dot{v} = h\gamma\left(u - P(h)\right) + 4(1-\gamma)\left(u - H(h)\right), \tag{11.3}$$
$$\dot{\gamma} = \beta_2\epsilon^2\gamma(1-\gamma)^2 Q(h). \tag{11.4}$$

Given η_4 with $0 < \eta_4 < 1$, we assume $\eta_4 \leq \frac{\tilde{C}_2}{\tilde{\beta}_2^5} \leq \eta_4^{-1}$ and $\tilde{\beta}_2$ is small. Therefore $\eta_4^{\frac{1}{2}} \leq \beta_2 \leq \eta_4^{-\frac{1}{2}}$, and $\epsilon = \frac{\tilde{\beta}_2}{\beta_2}$ is small. Hence (11.1)–(11.4) is a slow-fast system with two slow variables (h and Γ) and two fast variables (u and v).

For $\epsilon = 0$, the set

$$u = \frac{h\gamma P(h) + 4(1-\gamma)H(h)}{h\gamma + 4(1-\gamma)}, \quad v = 0 \tag{11.5}$$

is a normally hyperbolic manifold of equilibria of dimension 2 (one positive eigenvalue, one negative eigenvalue); compare (9.13). For small $\epsilon > 0$, the perturbed normally hyperbolic invariant manifold is given by

$$u = K(h,\gamma,\epsilon,\beta_2), \quad K(h,\gamma,0,\beta_2) = \frac{h\gamma P(h) + 4(1-\gamma)H(h)}{h\gamma + 4(1-\gamma)}, \tag{11.6}$$
$$v = L(h,\gamma,\epsilon,\beta_2), \quad L(h,\gamma,0,\beta_2) = 0; \tag{11.7}$$

compare (9.14)–(9.15).

The vector field on the perturbed normally hyperbolic invariant manifold is

$$\dot{h} = \epsilon\big(h\gamma P(h) + 4(1-\gamma)H(h) + O(\epsilon)\big), \tag{11.8}$$
$$\dot{\gamma} = \beta_2\epsilon^2\gamma(1-\gamma)^2 Q(h). \tag{11.9}$$

In slow time (i.e., divide by ϵ):

$$h' = h\gamma P(h) + 4(1-\gamma)H(h) + O(\epsilon), \tag{11.10}$$
$$\gamma' = \beta_2\epsilon\gamma(1-\gamma)^2 Q(h). \tag{11.11}$$

In these coordinates, for $\epsilon = 0$, each line $\gamma =$ constant is invariant and has equilibria at points h with $h\gamma P(h) + 4(1-\gamma)H(h) = 0$. By Proposition 5.1, the equilibria constitute two one-dimensional normally hyperbolic invariant manifolds

$$C_i = \{(h,\gamma) : h = h_i(\gamma, 4(1-\gamma)),\, 0 \le \gamma \le 1\}, \quad i = 1,\, 2.$$

See Figure 11.1. Equilibria on C_1 are repellers; equilibria on C_2 are attractors.

For small $\epsilon > 0$, C_1 (respectively C_2) perturbs to a normally hyperbolic invariant curve connecting $(h_L, 0, 0)$ to $(h_1, 0, 0)$ (respectively $(h_R, 0, 0)$ to $(h_2, 0, 0)$). On the perturbed normally hyperbolic invariant manifolds corresponding to C_i, we have

$$\gamma' = \beta_2 \epsilon \gamma (1-\gamma)^2 Q(h_i(\gamma, 4(1-\gamma)) + O(\beta_2)),$$

which is positive on the interior of C_1 and negative on the interior of C_2. There is a repeller at h_L and an attractor at h_R.

Singular solutions ($\epsilon = 0$) from $(h_L, 0)$ to $(h_R, 0)$ consist of a slow solution on C_1 from $(h_L, 0)$ to level γ, $0 < \gamma < 1$; a fast connection from C_1 to C_2 at level γ; and a slow solution on C_2 to $(h_R, 0)$. They persist for small $\epsilon > 0$. See Figure 11.1. □

As in Region 3, the slopes of C_1 and C_2 are positive where they meet $(h_L, 0)$ and $(h_R, 0)$ respectively, which produces, at the left, a rise in the traveling wave above h_L, and, at the right, a slow descent toward h_R.

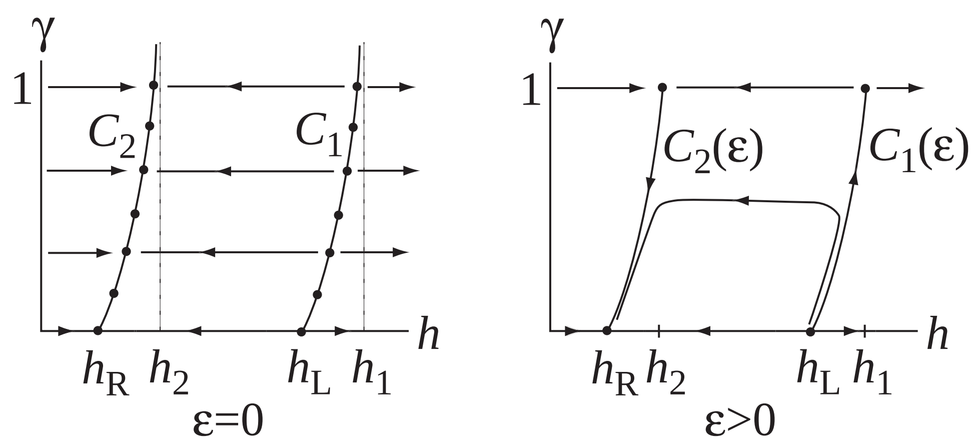


FIGURE 11.1

12. REGION 5

Theorem 12.1. *Let h_L and h_R satisfying* (3.15) *be given. Consider the traveling wave system* (3.11)–(3.14) *with the normalization (N3): $\tilde{C} = \tilde{C}_3 \ge 0$, $\tilde{D} = \tilde{D}_3 \ge 0$, and $\tilde{\beta} = 1$. Then for each small $\eta > 0$, there exists $\delta_5 > 0$ such that, if Region 5 is defined using this value, then for each parameter triple $(\tilde{C}_3, \tilde{D}_3, 1)$ in Region 5, there exists a one-parameter family of traveling waves connecting h_L to h_R. Moreover,* $\max \Gamma$ *ranges from 0 to η^{-1}.*

If we decrease η, then apparently δ_5 must shrink. Thus, unlike in Regions 2 and 4, we are not able to define a single Region 5 in which connecting orbits of all sizes exist.

To prove the theorem, in the traveling wave system (3.11)–(3.14) let

$$\epsilon = \left(\frac{\tilde{C}}{\tilde{\beta}^3}\right)^{\frac{1}{2}}, \quad \hat{v} = \epsilon v.$$

After dropping the hat and multiplying by ϵ, we obtain

$$\dot{h} = \epsilon(h\Gamma + 4\tilde{D})u, \tag{12.1}$$
$$\dot{u} = (h\Gamma + 4\tilde{D})v, \tag{12.2}$$
$$\dot{v} = h\Gamma\,(u - P(h)) + 4\tilde{D}\,(u - H(h))\,, \tag{12.3}$$
$$\dot{\Gamma} = \tilde{\beta}\epsilon\Gamma Q(h). \tag{12.4}$$

In this system let

$$\sigma = h\Gamma + 4\tilde{D}.$$

We obtain

$$\dot{h} = \epsilon\sigma u, \tag{12.5}$$
$$\dot{u} = \sigma v, \tag{12.6}$$
$$\dot{v} = (\sigma - 4\tilde{D})\,(u - P(h)) + 4\tilde{D}\,(u - H(h))\,, \tag{12.7}$$
$$\dot{\sigma} = \frac{\epsilon(\sigma - 4\tilde{D})}{h}\left(\sigma u + \tilde{\beta}hQ(h)\right). \tag{12.8}$$

We use the normalized parameters (N3), and we set

$$\tilde{D}_3 = \sigma D_3 \text{ and } \epsilon = \epsilon_3\sigma D_3.$$

Then after division by σ, the system (12.5)–(12.8) becomes

$$\dot{h} = \epsilon_3\sigma D_3 u, \tag{12.9}$$
$$\dot{u} = v, \tag{12.10}$$
$$\dot{v} = (1 - 4D_3)\,(u - P(h)) + 4D_3\,(u - H(h))\,, \tag{12.11}$$
$$\dot{\sigma} = \frac{\epsilon_3\sigma D_3(1 - 4D_3)}{h}\,(\sigma u + hQ(h))\,, \tag{12.12}$$
$$\dot{D}_3 = -\frac{\epsilon_3 D_3^2(1 - 4D_3)}{h}\,(\sigma u + hQ(h))\,. \tag{12.13}$$

Notice that (12.5)–(12.8), with the normalized parameters (N3), has four variables and the two parameters $\epsilon = \tilde{C}_3^{\frac{1}{2}}$ and $\tilde{D}_3$, while (12.9)–(12.13) has five variable and just one parameter, $\epsilon_3 = \frac{\epsilon}{\tilde{D}_3} = \left(\frac{\tilde{C}_3}{\tilde{D}_3^2}\right)^{\frac{1}{2}}$. Corresponding to the lost parameter, the new system has a constant of the motion: the product σD_3. To study Region 5, we assume $\frac{\tilde{C}_3}{\tilde{D}_3^2}$ and $\tilde{D}_3$ are small. Therefore the parameter ϵ_3 is small, and the product σD_3 is small.

For $D_3 = 0$ or $\epsilon_3 = 0$, the set

$$u = (1 - 4D_3)P(h) + 4D_3H(h), \quad v = 0$$

is a normally hyperbolic manifold of equilibria of dimension 2. (One positive eigenvalue, one negative eigenvalue.) The perturbed normally hyperbolic invariant manifold is

$$u = (1 - 4D_3)P(h) + 4D_3H(h) + O(\epsilon_3 D_3), \quad v = O(\epsilon_3 D_3).$$

On it, after division by $\epsilon_3 D_3$ we obtain

$$\dot{h} = \sigma\big((1-4D_3)P(h) + 4D_3H(h) + O(\epsilon_3 D_3)\big), \tag{12.14}$$

$$\dot{\sigma} = \frac{\sigma(1-4D_3)}{h}\left(\sigma\big((1-4D_3)P(h) + 4D_3H(h) + O(\epsilon_3 D_3)\big) + hQ(h)\right), \tag{12.15}$$

$$\dot{D}_3 = -\frac{D_3(1-4D_3)}{h}\left(\sigma\big((1-4D_3)P(h) + 4D_3H(h) + O(\epsilon_3 D_3)\big) + hQ(h)\right). \tag{12.16}$$

The $O(\epsilon_3 D_3)$ terms are all equal.

On the invariant plane $\sigma = 0$, the system (12.14)–(12.16) reduces to

$$\dot{h} = 0, \quad \dot{D}_3 = -D_3(1-4D_3)Q(h). \tag{12.17}$$

On the invariant plane $D_3 = 0$, the system (12.14)–(12.16) reduces to

$$\dot{h} = \sigma P(h), \quad \dot{\sigma} = \frac{\sigma}{h}\big(\sigma P(h) + hQ(h)\big). \tag{12.18}$$

See Figure 12.1(a). We remark that the integral curve of (12.18) that approaches $(h_a, 0)$ as $t \to -\infty$, $h_* < h_a < h_1$, is given by $\sigma = h\Phi(h_a, h)$,

$$\Phi(h_a, h) = \ln\left(\frac{h_1 - h_a}{h_1 - h}\right)^A \left(\frac{h - h_2}{h_a - h_2}\right)^B \left(\frac{h + h_1 + h_2}{h_a + h_1 + h_2}\right)^C, \tag{12.19}$$

$$A = \frac{2s(h_1 - h_*)h_1}{(h_1 - h_2)(2h_1 + h_2)}, \quad B = \frac{2s(h_* - h_2)h_2}{(h_1 - h_2)(2h_1 + h_2)}, \quad C = \frac{2s(h_1 + h_2 + h_*)(h_1 - h_2)}{h_1 h_2}. \tag{12.20}$$

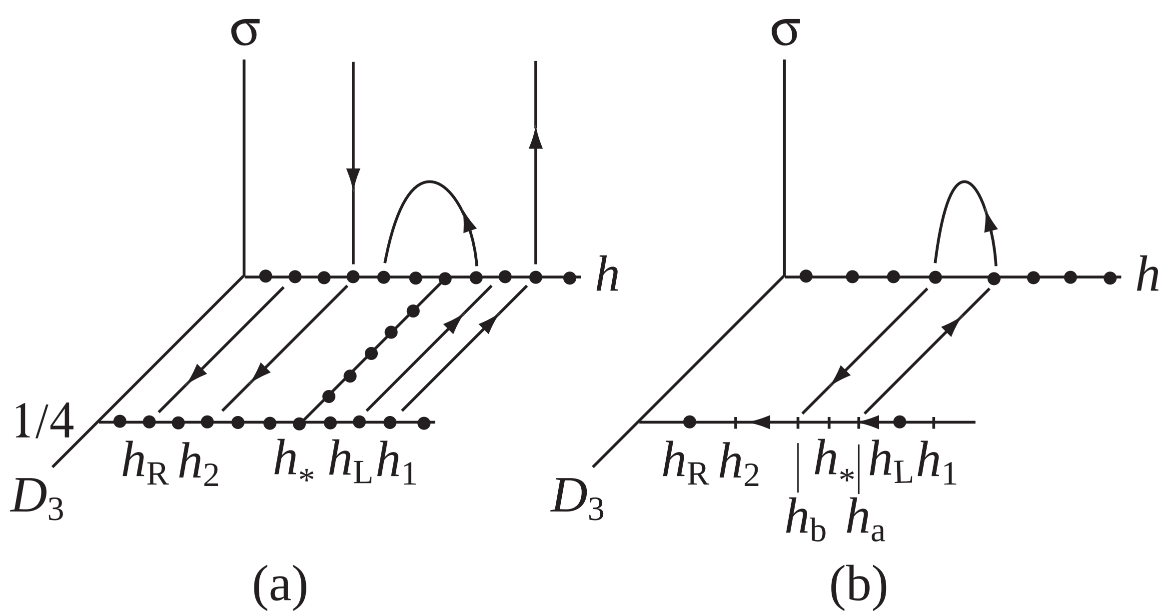


FIGURE 12.1. (a) Flow of (12.14)–(12.16) in the planes $\sigma = 0$ and $D_3 = 0$, for any $\epsilon_3 \geq 0$. (b) A singular solution. When finding points in this figure that correspond to points mentioned in the text, recall that points in the text are given as (h, σ, D_3).

On the invariant plane $D_3 = \frac{1}{4}$, the system (12.14)–(12.16) reduces to $\dot{h} = \sigma(H(h)+O(\epsilon_3))$, $\dot{\sigma} = 0$. Because of the known equilibria, the $O(\epsilon_3)$ term must be zero for $h = h_L$ and h_R. For small σ and ϵ_3, the equilibrium $(h_L, \sigma, \frac{1}{4})$ (respectively $(h_R, \sigma, \frac{1}{4})$) has two positive (respectively negative) eigenvalues.

Singular solution:

(1) In the plane $\sigma = 0$, from $(h_L, 0, \frac{1}{4})$, follow the "slow equation" $\dot{h} = H(h)$ along the line $D_3 = \frac{1}{4}$ to a point $(h_a, 0, \frac{1}{4})$ with $h_* < h_a < h_1$.
(2) In the plane $\sigma = 0$, follow a solution of (12.17), i.e., the line $h = h_a$, from $(h_a, 0, \frac{1}{4})$ to $(h_a, 0, 0)$.
(3) In the plane $D_3 = 0$, follow a solution of (12.18) from $(h_a, 0, 0)$ to a point $(h_b, 0, 0)$. We will necessarily have $h_2 < h_b < h_*$.
(4) In the plane $\sigma = 0$, follow the line $h = h_b$ from $(h_b, 0, 0)$ to $(h_b, 0, \frac{1}{4})$.
(5) In the plane $\sigma = 0$, from $(h_b, 0, \frac{1}{4})$, follow the "slow equation" $\dot{h} = H(h)$ along the line $D_3 = \frac{1}{4}$ to $(h_R, 0, \frac{1}{4})$.

See Figure 12.1(b).

Each set $\sigma D_3 =$ constant is invariant under (12.14)–(12.16); in fact, $\sigma D_3 = \tilde{D}_3$, which is constant on solutions. On it there are two equilibria, $(h_L, 4\tilde{D}_3, \frac{1}{4})$ and $(h_R, 4\tilde{D}_3, \frac{1}{4})$. See Figure 12.2.

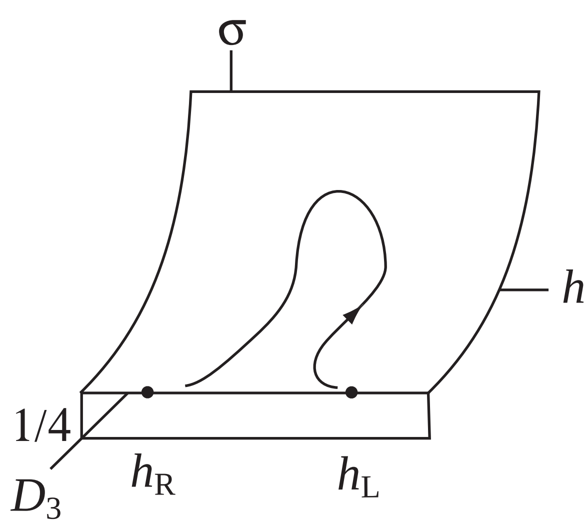


FIGURE 12.2. Flow of (12.14)–(12.16) on the surface $\sigma D_3 = \tilde{D}_3$ for any $\epsilon_3 \geq 0$, for $0 < \tilde{D}_3 << 1$ and for $0 < \epsilon_3 D_3 << 1$. When finding points in this figure that correspond to points mentioned in the text, recall that points in the text are given as (h, σ, D_3).

Proposition 12.2. *Given a singular solution, for $\tilde{D}_3 > 0$ and $\epsilon_3 \geq 0$ both sufficiently small, there is a solution of (12.14)–(12.16) on the surface $\sigma D_3 = \tilde{D}_3$ that is close to the singular solution.*

The asymptotic expansion of the solutions considered here will be presented elsewhere.

Note that in Figure 12.2, once one connection exists, the part of the surface inside it and with $\tilde{D}_3 < \frac{1}{4}$ is completely filled by connections. Thus Proposition 12.2 implies that corresponding to a fixed singular solution, there is a number $\delta_5 > 0$ such that if $0 < \tilde{D}_3 \le \delta_5$ and $0 \le \epsilon_3 \le \delta_5^{\frac{1}{2}}$, then connecting orbits near and inside the singular solution exist on the surface $\sigma D_3 = \tilde{D}_3$. Now $\tilde{C}_3 = \epsilon^2 = \epsilon_3^2(\sigma D_3)^2 \le \delta_5 \tilde{D}_3^2$. Thus Proposition 12.2 implies Theorem 12.1.

The slow parts of the singular solution (the first and fifth parts) give rise to "feet" in the connecting orbits.

The proof of Proposition 12.2 uses the following lemma.

Fix a small $\delta > 0$, unrelated to previous δ's, and let $I \subset (h_*, h_1)$ be a closed interval. For $h_0 \in I$, $\tilde{D}_3 \ge 0$ and $\epsilon_3 \ge 0$ small, define $\phi(h_0, \tilde{D}_3, \epsilon_3)$ as follows (see Figure 12.3):

(1) If $\tilde{D}_3 > 0$, follow the solution of (12.14)–(12.16), for the given value of ϵ_3, in the surface $\sigma D_3 = \tilde{D}_3$, that starts at $(h, \sigma, D_3) = (h_0, \delta, \frac{\tilde{D}_3}{\delta})$, until it reaches a point $(h, \frac{\tilde{D}_3}{\delta}, \delta)$. Then $\phi(h_0, \tilde{D}_3, \epsilon_3) = h$.

(2) For the system (12.14)–(12.16) and $h_0 \in I$, let the unstable manifold of the equilibrium $(h, \sigma, D_3) = (h_0, 0, 0)$, a curve in the plane $D_3 = 0$, meet the plane $\sigma = \delta$ at the point $(h, \delta, 0)$. Then $\phi(h_0, 0, \epsilon_3) = h$.

The mapping ϕ is smooth on $\{(h_0, \tilde{D}_3, \epsilon_3) : \tilde{D}_3 > 0\}$, and its restriction to $\{(h_0, \tilde{D}_3, \epsilon_3) : \tilde{D}_3 = 0\}$ is smooth.

Let $\phi_{(\tilde{D}_3, \epsilon_3)}(h_0) = \phi(h_0, \tilde{D}_3, \epsilon_3)$.

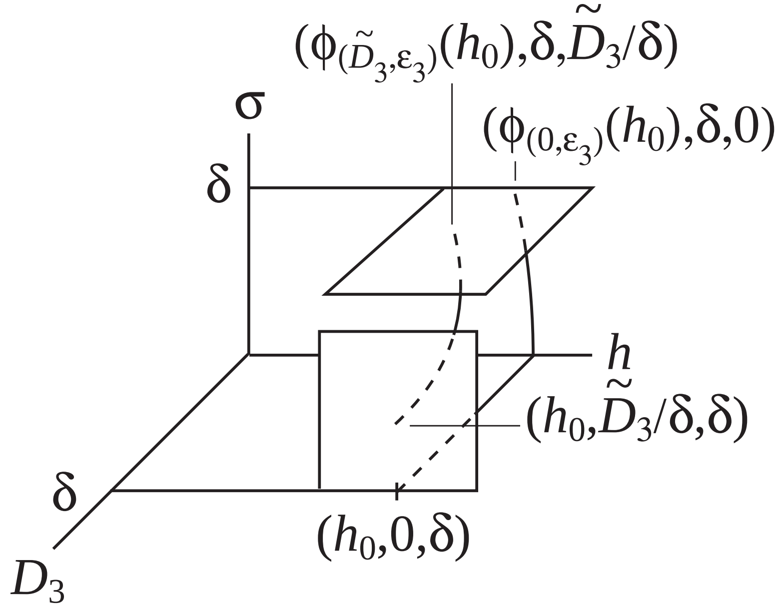


FIGURE 12.3. Definition of ϕ for fixed ϵ_3. Points are given as (h, σ, D_3).

Lemma 12.3. *The mapping* $(\tilde{D}_3, \epsilon_3) \to \phi_{(\tilde{D}_3, \epsilon_3)}$ *is continuous from a neighborhood of* $(0, 0)$ *in* $\{(\tilde{D}_3, \epsilon_3) : \tilde{D}_3 \ge 0$ *and* $\epsilon_3 \ge 0\}$ *to* $C^1(I, \mathbb{R})$.

We prove this lemma in Section 14. Given the lemma, the proof of Proposition 12.2 goes as follows. Fix a small $\epsilon_3 \ge 0$ and consider small $\tilde{D}_3 > 0$. We need ϵ_3 small so that the $O(\epsilon_3 D_3)$ term will be small for $0 < D_3 \le \frac{1}{4}$.

For the system (12.14)–(12.16), the set $I \times \{\frac{1}{4}\} \times \{0\}$, regarded as a subset of $\sigma = 0$, is a normally hyperbolic (repelling) line of equilibria. Therefore the invariant line $I\times\{\frac{1}{4}\}\times\{4\tilde{D}_3\}$, regarded as a subset of the invariant surface $\sigma D_3 = \tilde{D}_3$, is a normally hyperbolic (repelling) invariant line. It contains one equilibrium, $(h_L, \frac{1}{4}, 4\tilde{D}_3)$, which is a repeller within the line $I \times \{\frac{1}{4}\} \times \{4\tilde{D}_3\}$, with eigenvalue close to 0 when $\tilde{D}_3$ is small.

Consider the solution of (12.14)–(12.16) in $\sigma D_3 = \tilde{D}_3$ that starts at $(h, \sigma, D_3) = (h_a, \frac{4\tilde{D}_3}{1-\delta}, \frac{1-\delta}{4})$. From the previous paragraph, it follows that in backward time this solution approaches the equilibrium $(h_L, 4\tilde{D}_3, \frac{1}{4})$. Moreover, as $\tilde{D}_3 \to 0$, the solution curve approaches the union of the curves $\{(h_a, 0, D_3) : \frac{1}{4} \leq D_3 \leq \frac{1-\delta}{4}\}$ and $\{(h, 0, \frac{1}{4}) : h$ between h_L and $h_a\}$.

In forward time the solution arrives at the plane $D_3 = \delta$ at a point $(h_0, \frac{\tilde{D}_3}{\delta}, \delta)$ with $h_0 \to h_a$ as $\tilde{D}_3 \to 0$. The solution then arrives at the plane $\sigma = \delta$ at $(\phi_{(\tilde{D}_3,\epsilon_3)}(h_0), \delta, \frac{\tilde{D}_3}{\delta})$.

For $h \in I$, the unstable manifold of $(h, 0, 0)$, which lies in the plane $D_3 = 0$, meets the plane $\sigma = \delta$ at $\phi_{(0,\epsilon_3)}(h), \delta, 0)$, which is independent of ϵ_3. By Lemma 12.3, $\phi_{(\tilde{D}_3,\epsilon_3)}(h_0)$ is close to $\phi_{(0,\epsilon_3)}(h_0)$, and of course $\phi_{(0,\epsilon_3)}(h_0)$ is close to $\phi_{(0,\epsilon_3)}(h_a)$.

The remainder of the proof consists of continuing to follow this solution using similar arguments until it arrives at $(h_R, \frac{1}{4}, 4\tilde{D}_3)$.

13. Region 6

Theorem 13.1. *Let h_L and h_R satisfying* (3.15) *be given. Consider the traveling wave system* (3.11)–(3.14) *with the normalization (N3): $\tilde{C} = \tilde{C}_3 \geq 0$, $\tilde{D} = \tilde{D}_3 \geq 0$, and $\tilde{\beta} = 1$. Then for each η with $0 < \eta < 1$, there exists $\delta_6 > 0$ such that, if Region 6 is defined using this value, then for each parameter triple $(\tilde{C}_3, \tilde{D}_3, 1)$ in Region 6, there exists a one-parameter family of traveling waves connecting h_L to h_R. Moreover,* $\max\Gamma$ *ranges from η to η^{-1}.*

If we decrease η, then apparently δ_6 must shrink. Thus, as in Region 5, we are not able to define a single Region 6 in which connecting orbits of all sizes exist. Also note that Theorem 13.1 does not say anything about connecting orbits with $\max\Gamma$ close to 0. Section 7 provides some information about these.

To prove the theorem, in (12.1)–(12.4), a system with four variables and the three parameters $\epsilon = \left(\frac{\tilde{C}}{\tilde{\beta}^3}\right)^{\frac{1}{2}}$, $\tilde{D}$, and $\tilde{\beta}$, we use the normalization (N3), obtaining a system with four variables and the two parameters $\epsilon = \tilde{C}_3^{\frac{1}{2}}$ and $\tilde{D}_3$. We then let

$$\Gamma = r(1-\theta),\ \epsilon = r\theta,\ \tilde{D}_3 = D_3\epsilon = D_3 r\theta.$$

After division by r, (12.1)–(12.4) becomes

$$\dot{h} = r\theta(h(1-\theta) + 4D_3\theta)u, \tag{13.1}$$

$$\dot{u} = (h(1-\theta) + 4D_3\theta)v, \tag{13.2}$$

$$\dot{v} = h(1-\theta)\left(u - P(h)\right) + 4D_3\theta\left(u - H(h)\right), \tag{13.3}$$

$$\dot{r} = r\theta(1-\theta)Q(h), \tag{13.4}$$

$$\dot{\theta} = -\theta^2(1-\theta)Q(h). \tag{13.5}$$

This is a system with five variables and just one parameter, $D_3 = \frac{\tilde{D}_3}{\epsilon} = \frac{\tilde{D}_3}{\tilde{C}_3^{\frac{1}{2}}}$. Corresponding to the lost parameter, the new system has a constant of the motion: the product $r\theta$. To study Region 6 we take D_3 and $\tilde{C}_3$ to be small. Since $\tilde{C}_3$ is small, ϵ is small, so the product $r\theta$ is small.

The system (13.1)–(13.5) has, for any D_3, two curves of equilibria with $\theta = 1$:

$$A = \{(h_L, 0, 0, r, 1) : r \geq 0\} \text{ and } B = \{(h_R, 0, 0, r, 1) : r \geq 0\}.$$

For $D_3 > 0$ and $r > 0$ these equilibria have three-dimensional unstable and stable manifolds respectively. We wish to study intersections of these manifolds.

Our analysis will proceed as follows.

1. In the region $0 \leq \theta < 1$ we identify, for any D_3, a three-dimensional manifold M, parameterized by (h, r, θ), that is normally hyperbolic, with one-dimensional stable fibers and one-dimensional unstable fibers. On M, after a rescaling, the flow is like that in Figure 12.1 of the previous section; see Figure 13.1 below. However, in the previous section, the invariant manifold extended to $D_3 = \frac{1}{4}$, where the equilibria at $h = h_L$ and $h = h_R$ lay. Here M does not extend to $\theta = 1$, where the curves of equilibria A and B lie.

2. We show that the families of unstable and stable manifolds of equilibria in A and B respectively extend smoothly to $r = 0$ and $D_3 = 0$.

3. Within $r = 0$, for $D_3 = 0$, $W^s(M)$ and $W^u(M)$ do extend smoothly to $\theta = 1$. We show that within $r = 0$, for $D_3 = 0$, $W^u(A)$ meets $W^s(M)$ transversally, and $W^s(B)$ meets $W^u(M)$ transversally. It follows that the four-dimensional manifold $W^u(A)$ meets the four-dimensional manifold $W^s(M)$ transversally, and the four-dimensional manifold $W^s(B)$ meets the four-dimensional manifold $W^u(M)$ transversally.

4. We track $W^u(A)$ in forward time as it passes the line of equilibria in $r = \theta = 0$ in M. Using a "corner lemma" to be proved in Section 15, we show that after passing the line of equilibria, $W^u(A)$ is C^1-close to $W^u(M)$. Similarly, we track $W^s(B)$ in backward time. After passing the line of equilibria, $W^s(B)$ is C^1-close to $W^s(M)$.

5. It follows that for each fixed D_3 near 0, $W^u(A)$ and $W^s(B)$ meet transversally in a three-dimensional manifold. Past the line of equilibria, i.e., for r of order 1, this manifold is C^1-close to M. It is therefore foliated into two-dimensional invariant manifolds parameterized by small $\tilde{C}_3$: $r\theta = \epsilon = \tilde{C}_3^{\frac{1}{2}}$. Each manifold is filled with connecting orbits of (13.1)–(13.5) from $(h_L, 0, 0, \epsilon, 1)$ to $(h_R, 0, 0, \epsilon, 1)$.

Somewhat more precisely, in Figure 13.1 below, consider the solution curves in $\theta = 0$ with $\frac{\eta}{2} \leq \max r \leq 2\eta^{-1}$. They limit in backward time on points (h, r, θ) with $r = \theta = 0$ and $h \in I$, where I is a closed interval in (h_*, h_1). Let $\hat{I}$ be a closed interval contained in the interior of I. Then there is a number $\delta_6 > 0$ such that if $0 < D_3 < \delta$ and $0 < \tilde{C}_3 < \delta$, then $W^u(A) \cap W^s(B)$ includes a 2-dimensional invariant manifold in $r\theta = \epsilon = \tilde{C}_3^{\frac{1}{2}}$ that sweeps past the portion of the line of equilibria that has $h \in \hat{I}$. If we use δ_6 to define Region 6, then for each triple of parameters in Region 6, (13.1)–(13.5) has a one-parameter family of connecting orbits from $(h_L, 0, 0, 0, 0)$ to $(h_R, 0, 0, 0, 0)$ as described in the theorem.

In the next four subsections we give more details for steps 1–4 in the analysis.

13.1. **The invariant manifold** M.

For any D_3, the system (13.1)–(13.5) has the two-dimensional manifold of equilibria

$$u = P(h), \quad v = 0, \quad \theta = 0, \quad h > 0, \quad r \text{ arbitrary.} \tag{13.6}$$

Normal to this manifold, each equilibrium has one positive eigenvalue, one negative eigenvalue, and one zero eigenvalue.

The usual proof of the Center Manifold Theorem shows that the manifold (13.6) is part of a three-dimensional normally hyperbolic invariant manifold M in $huvr\theta$-space; in addition, M depends on the parameter D_3. It is given by

$$u = K(h, r, \theta, D_3), \quad K(h, r, 0, D_3) = P(h), \tag{13.7}$$

$$v = L(h, r, \theta, D_3), \quad L(h, r, 0, D_3) = 0. \tag{13.8}$$

K and L are initially only defined for θ small. However, M and its stable and unstable manifolds can be extended by the flow. In particular, within the invariant space $r = 0$, for $D_3 = 0$, (13.1)–(13.5) reduces to

$$\dot{h} = 0, \tag{13.9}$$

$$\dot{u} = h(1-\theta)v, \tag{13.10}$$

$$\dot{v} = h(1-\theta)\left(u - P(h)\right), \tag{13.11}$$

$$\dot{\theta} = -\theta^2(1-\theta)Q(h). \tag{13.12}$$

Divide by $h(1-\theta)$:

$$\dot{h} = 0, \tag{13.13}$$

$$\dot{u} = v, \tag{13.14}$$

$$\dot{v} = u - P(h), \tag{13.15}$$

$$\dot{\theta} = -\frac{1}{h}\theta^2 Q(h). \tag{13.16}$$

From the form (13.13)–(13.16) one easily sees that within the invariant space $r = 0$, for $D_3 = 0$, M extends in the positive θ-direction to

$$u = P(h), \quad v = 0, \quad h > 0, \quad 0 \le \theta < 1. \tag{13.17}$$

Note that in (13.5), $\dot{\theta} = 0$ for $\theta = 1$, so M cannot be extended by the flow to $\theta = 1$.

Within the invariant space $r = 0$, for $D_3 = 0$, $W^u(M)$ is given by

$$v = u - P(h), \quad h > 0, \quad u \text{ arbitrary}, \quad 0 \le \theta < 1, \tag{13.18}$$

and $W^s(M)$ is given by

$$v = -u + P(h), \quad h > 0, \quad u \text{ arbitrary}, \quad 0 \le \theta < 1. \tag{13.19}$$

Note that *within the invariant space* $r = 0$, *for* $D_3 = 0$, M, $W^u(M)$, and $W^s(M)$ can be extended smoothly to $\theta = 1$. However, normal hyperbolicity of M in the system (13.1)–(13.5) is certainly lost there. We do not attempt to extend M, $W^u(M)$, or $W^s(M)$ to $\theta = 1$ outside $r = 0$, $D_3 = 0$.

On M, after division by θ, (13.1)–(13.5) reduces to

$$\begin{aligned}\dot{h} &= r(h(1-\theta) + 4D_3\theta)K(h, r, \theta, D_3) \\ &= r(h(1-\theta) + 4D_3\theta)\big(P(h) + O(\theta(|r| + |D_3|))\big),\end{aligned} \tag{13.20}$$

$$\dot{r} = r(1-\theta)Q(h), \tag{13.21}$$

$$\dot{\theta} = -\theta(1-\theta)Q(h). \tag{13.22}$$

Each surface $r\theta = \epsilon$ is invariant. See Figure 13.1. We remark that for $\theta = 0$, the integral curve of (13.20)–(13.21) that approaches $(h_a, 0)$ as $t \to -\infty$, $h_* < h_a < h_1$, is given by $r = \Phi(h_a, h)$, where $\Phi(h_a, h)$ is given by (12.19)–(12.20).

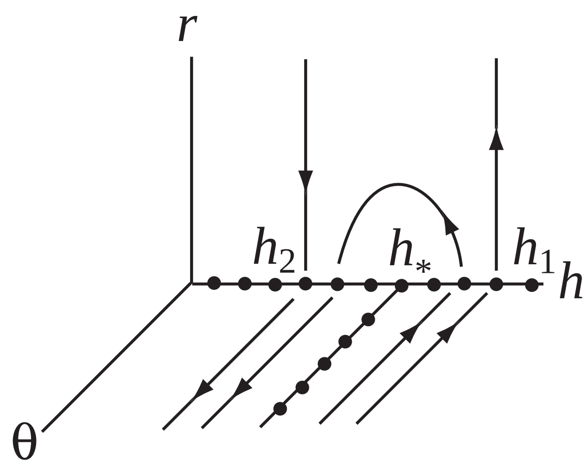


FIGURE 13.1. Flow of (13.20)–(13.22) on the 3-dimensional normally hyperbolic invariant manifold M.

13.2. **The curves of equilibria** A **and** B**.** For $D_3 = 0$, the space $\theta = 1$ consists of equilibria of (13.1)–(13.5). It is normally hyperbolic away from $h = h_*$. The space $\theta = 1$ remains invariant for $D_3 > 0$. For D_3 small it is normally hyperbolic away from $h = h_*$.

Restricting (13.1)–(13.5) to $\theta = 1$ and dividing by $4D_3$, we obtain

$$\dot{h} = ru, \tag{13.23}$$
$$\dot{u} = v, \tag{13.24}$$
$$\dot{v} = u - H(h), \tag{13.25}$$
$$\dot{r} = 0. \tag{13.26}$$

Restrict to the invariant space $r = 0$:

$$\dot{h} = 0, \tag{13.27}$$
$$\dot{u} = v, \tag{13.28}$$
$$\dot{v} = u - H(h). \tag{13.29}$$

The system (13.27)–(13.29) has, for each D_3, the curve of equilibria

$$u = H(h), \quad v = 0, \quad h > 0. \tag{13.30}$$

It is normally hyperbolic (one positive eigenvalue, one negative eigenvalue). Unstable manifold:

$$v = u - H(h), \quad h > 0, \quad u \text{ arbitrary}. \tag{13.31}$$

Stable manifold:

$$v = -u + H(h), \quad h > 0, \quad u \text{ arbitrary}. \tag{13.32}$$

For each D_3, the curve (13.30) is part of a two-dimensional normally hyperbolic invariant manifold N in $\theta = 1$ for the system (13.23)–(13.26). N is given by

$$u = \hat{K}(h, r, D_3), \quad \hat{K}(h, 0, D_3) = H(h), \tag{13.33}$$
$$v = \hat{L}(h, r, D_3), \quad \hat{L}(h, 0, D_3) = 0. \tag{13.34}$$

In these expressions, $r \geq 0$ and $D_3 \geq 0$ are small. We must have

$$\hat{K}(h_L, r, D_3) = \hat{L}(h_L, r, D_3) = \hat{K}(h_R, r, D_3) = \hat{L}(h_R, r, D_3) = 0.$$

On N the system (13.23)–(13.26) reduces to

$$\dot{h} = r\hat{K}(h, r, D_3), \tag{13.35}$$

$$\dot{r} = 0. \tag{13.36}$$

Divide by r to desingularize at $r = 0$:

$$\dot{h} = \hat{K}(h, r, D_3), \tag{13.37}$$

$$\dot{r} = 0. \tag{13.38}$$

For fixed $D_3 \geq 0$, regard A as a curve in the four-dimensional space $\theta = 1$. It lies in the two-dimensional manifold N, where it is a curve of normally repelling equilibria of (13.37)–(13.38). Let $\tilde{A}$ be a small neighborhood of A in N; $\tilde{A}$ has dimension two. Since N is normally hyperbolic in $\theta = 1$ for the system (13.23)–(13.26), each point of N has a one-dimensional unstable fiber for this system. Let $W^u(\tilde{A})$ denote the union over points of $\tilde{A}$ of these unstable fibers, a three-dimensional submanifold of $\theta = 1$. Now for h near h_L, the space $\theta = 1$ is a normally repelling invariant manifold for the system (13.1)–(13.5). Therefore each point of $W^u(\tilde{A})$ has a one-dimensional unstable fiber for this system. Let $W^u(A)$ denote the union over points of $W^u(\tilde{A})$ of these unstable fibers, a four-dimensional manifold. $W^u(A)$ is given by

$$v = \hat{L}_u(h, u, r, \theta, D_3), \quad \hat{L}_u(h, u, 0, 1, D_3) = u - H(h), \quad h \text{ near } h_L. \tag{13.39}$$

The notation $W^u(A)$ involves some abuse of notation when $D_3 = 0$ or $r = 0$. For $D_3 > 0$ and $r > 0$, points of $W^u(A)$ lie in the unstable manifold of one of the equilibria in A. This construction shows how these manifolds extend smoothly to $D_3 = 0$ and $r = 0$.

Define $W^s(B)$ similarly. It is given by

$$v = \hat{L}_s(h, u, r, \theta, D_3), \quad \hat{L}_s(h, u, 0, 1, D_3) = -u + H(h), \quad h \text{ near } h_R. \tag{13.40}$$

13.3. **Transversality.** We claim that for $D_3 \geq 0$ small, the four-dimensional manifolds $W^u(A)$, given by (13.39), and $W^s(M)$, given within $r = 0$ for $D_3 = 0$ by (13.19), are transverse.

To see this, set $D_3 = 0$ and look within the space $r = 0$, $\theta = 1$. There $W^u(A)$ is given by $v = u - H(h)$, $h > h_*$, u arbitrary; and $W^s(M)$ within $r = 0$, for $D_3 = 0$, extends smoothly to a submanifold of $r = 0$, $\theta = 1$ given by $v = -u + P(h)$, $h > 0$ arbitrary, u arbitrary. The transversal intersection is

$$u = \frac{1}{2}(H(h) + P(h)), \quad v = \frac{1}{2}(P(h) - H(h)), \quad h > h_*.$$

Similarly, for $D_3 = 0$, within the space $r = 0$, $W^s(B)$ is given by $v = -u + H(h)$, $h < h_*$, u arbitrary; and $W^u(M)$ within $r = 0$, for $D_3 = 0$, extends smoothly to a submanifold of $r = 0$, $\theta = 1$ given by $v = u - P(h)$, $h > 0$ arbitrary, u arbitrary. The transversal intersection is

$$u = \frac{1}{2}(H(h) + P(h)), \quad v = \frac{1}{2}(P(h) - H(h)), \quad 0 < h < h_*.$$

Let $N^s = W^u(A) \cap W^s(M)$, $N^u = W^s(B) \cap W^u(M)$. For fixed D_3, each is a three-dimensional manifold parameterized by (h, r, θ).

13.4. **Tracking solutions.** More precisely, for fixed D_3 and θ near 1, N^s is a section of $W^s(M)$, thought of as a bundle over M whose fibers are the one-dimensional stable fibers of points. N^s is thus a two-parameter family of solutions, parameterized by (h,r). The solutions track and approach solutions in M. For $r=0$, h (and of course r) are constant on solutions; the tracked solutions, and hence the corresponding solutions in N^s, approach $(h,0,0)$. Let I be a compact subinterval of (h_*,h_1) and let $\delta>0$ be small. Then for each D_3, N^s includes a curve J^s close to $\{(h,u,v,r,\theta): h\in I, r=0, \theta=\delta, u=K(h,0,\delta,D_3), v=L(h,0,\delta,D_3)\}$; each point of J^s is in the stable fiber of the corresponding point in M.

In order to track solutions farther, in (13.1)–(13.5) we let

$$u=K(h,r,\theta,D_3)+\tilde{u}, \quad v=L(h,r,\theta,D_3)+\tilde{v}.$$

In the new coordinates, $hr\theta$-space is M. We obtain

$$\dot{h}=r\theta(h(1-\theta)+4D_3\theta)(K(h,r,\theta,D_3)+\tilde{u}), \tag{13.41}$$
$$\dot{\tilde{u}}=O(|\tilde{u}|+|\tilde{v}|), \tag{13.42}$$
$$\dot{\tilde{v}}=O(|\tilde{u}|+|\tilde{v}|), \tag{13.43}$$
$$\dot{r}=r\theta(1-\theta)Q(h), \tag{13.44}$$
$$\dot{\theta}=-\theta^2(1-\theta)Q(h). \tag{13.45}$$

Next divide by $(1-\theta)Q(h)$, which is positive for $h\in I$ and θ near 0, and replace $(\tilde{u},\tilde{v})$ by new coordinates $(x,y)=(x,y)(h,\tilde{u},\tilde{v},r,\theta,D_3)$ in order to achieve Fenichel coordinates. We have

$$\dot{x}=-xc(x,y,h,r,\theta,D_3), \tag{13.46}$$
$$\dot{y}=yd(x,y,h,r,\theta,D_3), \tag{13.47}$$
$$\dot{h}=r\theta(a(h,\theta)+D_3\theta\, b(h,\theta))(K(h,r,\theta,D_3)+O(xy)), \tag{13.48}$$
$$\dot{r}=r\theta, \tag{13.49}$$
$$\dot{\theta}=-\theta^2, \tag{13.50}$$

with $c>0$, $d>0$, $a(h,\theta)=hQ^{-1}$, and $b(h,\theta)=4(1-\theta)^{-1}Q^{-1}$; a and b are positive for $h\in I$ and θ near 0. In Fenichel coordinates, $W^u(M)$ is given by $x=0$ and $W^s(M)$ is given by $y=0$; on both spaces, $(\dot{h},\dot{r},\dot{\theta})$ depends only on (h,r,θ).

In these coordinates,

$$J^s=\{(x,y,h,r,\theta): y=0, h\in I, r=0, \theta=\delta, x=\hat{x}(h,D_3)\}.$$

We extend J^s to a two-dimensional cross-section R^s of N^s,

$$R^s=\{(x,y,h,r,\theta): y=0, h\in I, 0\le r\le\delta_1, \theta=\delta, x=\hat{x}(h,r,D_3)\},$$

and we extend R^s to a three-dimensional cross-section Q^s of $W^u(A)$:

$$Q^s=\{(x,y,h,r,\theta): |y|\le\delta, h\in I, 0\le r\le\delta_1, \theta=\delta, x=\hat{x}(y,h,r,D_3)\}.$$

For (y^0,h^0,r^0,D_3) with $r^0>0$, follow the solution of (13.46)–(13.50), for the given value of D_3, that starts at $(x,y,h,r,\theta)=(\hat{x}(y^0,h^0,r^0,D_3),y^0,h^0,r^0,\delta)$ until it reaches $r=\delta$ at a point (x,y,h,δ,θ). The set of such points is a three-dimensional cross section Q^s_* of $W^u(A)$.

Proposition 13.2. *Let $\hat{I}$ be a closed interval contained in the interior of I. Then for $\delta > 0$ sufficiently small, there is a number $\delta_1 > 0$ such that if $0 \le D_3 \le \delta$, then Q^s_* contains $\{(x, y, h, r, \theta) : |y| \le \delta, h \in \hat{I}, r = \delta, 0 < \theta \le \delta_1, x = \tilde{x}(y, h, \theta, D_3)\}$. As $\theta \to 0$, $\tilde{x} \to 0$ in $C^1([-\delta, \delta] \times \hat{I} \times [0, \delta], \mathbb{R})$.*

This proposition implies that after passing the line of equilibria, $W^u(A)$ is C^1-close to $W^u(M)$. The proof is in Section 15.

14. Proof of Lemma 12.3

We write the system (12.14)–(12.16), with β fixed, as

$$\dot{h} = \sigma b(h, \sigma, D_3, \epsilon_3), \tag{14.1}$$

$$\dot{\sigma} = \sigma a(h, \sigma, D_3, \epsilon_3), \tag{14.2}$$

$$\dot{D}_3 = -D_3 a(h, \sigma, D_3, \epsilon_3). \tag{14.3}$$

For $h \in I$ and the other variables small, $a > 0$. Rescale by dividing by $a > 0$:

$$\dot{h} = \sigma \tilde{b}(h, \sigma, D_3, \epsilon_3), \tag{14.4}$$

$$\dot{\sigma} = \sigma, \tag{14.5}$$

$$\dot{D}_3 = -D_3. \tag{14.6}$$

Replacing h with $z = z(h, D_3)$, we can ensure that the unstable manifold of each equilibrium is a line $z =$ constant in the plane $D_3 = 0$. In these Fenichel coordinates [7] the system becomes:

$$\dot{z} = \sigma D_3\, c(z, \sigma, D_3, \epsilon_3), \tag{14.7}$$

$$\dot{\sigma} = \sigma, \tag{14.8}$$

$$\dot{D}_3 = -D_3. \tag{14.9}$$

Consider the following boundary value problem for (14.7)–(14.9):

$$z(0) = z^0, \quad \sigma(\tau) = \sigma^1, \quad D_3(0) = D_3^0, \quad \tau \ge 0.$$

Denote the solution $(z, \sigma, D_3)(t, \tau, z^0, \sigma^1, D_3^0, \epsilon_3)$. Fix $k > 0$. According to Deng's Lemma [4], for small $\delta > 0$ there are constants λ, μ, and K, with $-1 < \lambda < 0 < \mu < 1$ and $K > 0$, such that if $\max(|\sigma^1|, |D_3^0|, |\epsilon_3|) \le \delta$ and $z^0 \in I$, then the following is true. For each $i = 0, \dots, k$ and each $\mathbf{i}$-tuple of integers chosen from the set $\{1, \dots, 6\}$, $\|D^{\mathbf{i}}(z - z^0)\| \le K e^{\lambda t + \mu(t - \tau)}$.

Fix a small $\delta > 0$. In our coordinates ϕ corresponds to a mapping $\psi(z^0, D_3, \epsilon_3)$, with $z^0 \in I$ and $D_3 \ge 0$ and $\epsilon \ge 0$ small, defined as follows:

(1) If $D_3 > 0$, follow the solution of (14.7)–(14.9), for the given value of ϵ_3, in the surface $xy = D_3$, that starts at $(x, y, z) = (\delta, \frac{D_3}{\delta}, z^0)$, until it reaches a point $(x, y, z) = (\frac{D_3}{\delta}, \delta, z^1)$, which happens at time $\tau = \ln \frac{\delta^2}{D_3}$. Then $\phi(z^0, D_3, \epsilon_3) = z^1$.
(2) $\phi(z^0, 0, \epsilon_3) = z^0$.

For $D_3 > 0$ we have

$$z^1 = z(\tau, \tau, \delta, \delta, z^0) = z(\ln \frac{\delta^2}{D_3}, \ln \frac{\delta^2}{D_3}, \delta, \delta, z^0).$$

By Deng's Lemma

$$|z^1 - z^0| \le Ke^{\lambda\tau} = K\left(\frac{\delta^2}{D_3}\right)^{\lambda} = K\left(\frac{D_3}{\delta^2}\right)^{-\lambda}.$$

Since $\lambda < 0$, $z^1 - z^0 \to 0$ uniformly in z^0 as $D_3 \to 0$. Similarly, $\frac{\partial z^1}{\partial z^0} - 1 \to 0$ uniformly in z^0 as $D_3 \to 0$.

15. Proof of Proposition 13.2

Denote the solution of (13.48)–(13.50) with $x = 0$ or $y = 0$ and $(h, r, \theta)(0) = (h^1, r^1, \theta^1)$ by

$$(h_\sharp, r_\sharp, \theta_\sharp)(t, h^1, r^1, \theta^1, D_3), \quad r_\sharp = r^1(1 + \theta^1 t), \quad \theta_\sharp = \frac{\theta^1}{1 + \theta^1 t}.$$

Consider the following boundary value problem for (13.46)–(13.50):

$$x(0) = x^0, \quad (y, h, r, \theta)(\tau) = (y^1, h^1, r^1, \theta^1), \quad \tau \ge 0.$$

Denote the solution $(x, y, h, r, \theta)(t, \tau, x^0, y^1, h^1, r^1, \theta^1, D_3)$. Then

$$(r, \theta)(t, \tau, x^0, y^1, h^1, r^1, \theta^1, D_3) = (r_\sharp, \theta_\sharp)(t - \tau, h^1, r^1, \theta^1, D_3)$$

and

$$h(t, \tau, 0, y^1, h^1, r^1, \theta^1, D_3) = h(t, \tau, x^0, 0, h^1, r^1, \theta^1, D_3) = h_\sharp(t - \tau, h^1, r^1, \theta^1, D_3).$$

Moreover, according to Deng's Lemma [4], for small $\delta > 0$ there are constants λ, μ, and K, with $\lambda < 0 < \mu$ and $K > 0$, such that if $\max(|x^0|, |y^1|, |r^1|, |\theta^1|, |D_3|) \le \delta$ and $h^1 \in \hat{I}$, then :

$$\begin{aligned} &\|Dx\| \le Ke^{\lambda t}, \\ &\|Dy\| \le Ke^{\mu(t-\tau)}, \\ &\|D(h(t, \tau, x^0, y^1, h^1, r^1, \theta^1, D_3) - h_\sharp(t - \tau, h^1, r^1, \theta^1, D_3))\| \le Ke^{\lambda t + \mu(t-\tau)}. \end{aligned}$$

In addition, we note that for a small $\gamma > 0$, $\|D_{(h^1, r^1, \theta^1, D_3)}(h_\sharp, r_\sharp, \theta_\sharp)\| \le Ke^{\gamma|t|}$. These estimates hold as long as $\max(|x|, |y|, |r|, |\theta|, |D_3|) \le 2\delta$ and $h \in I$.

To prove Proposition 13.2, given $(y^1, h^1, r^1, \theta^1, D_3)$ with $|y^1| \le \delta$, $h^1 \in I$, $r^1 = \delta$, $\theta^1 > 0$ small, and D_3 small, we want to find (τ, x^0) such that

$$\theta(0, \tau, x^0, y^1, h^1, \delta, \theta^1, D_3) = \delta \text{ and } x^0 = \hat{x}((y, h, r)(0, \tau, x^0, y^1, h^1, \delta, \theta^1, D_3), D_3).$$

Then

$$\tilde{x}(y^1, h^1, \theta^1, D_3) = x(\tau, \tau, x^0, y^1, h^1, \delta, \theta^1, D_3). \tag{15.1}$$

Clearly

$$\tau = \frac{\delta - \theta^1}{\delta\theta^1}. \tag{15.2}$$

Let

$$x^0 = \hat{x}(0, (h_\sharp, r_\sharp)(-\tau, h^1, \delta, \theta^1, D_3), D_3) + \bar{x}^0 \text{ with } \tau = \frac{\delta - \theta^1}{\delta\theta^1}. \tag{15.3}$$

Define

$$G(\bar{x}^0, (h^1, \theta^1, D_3), y^1) = x^0 - \hat{x}((y, h, r)(0, \tau, x^0, y^1, h^1, \delta, \theta^1, D_3), D_3)$$

with τ and x^0 given in terms of $(\bar{x}^0, h^1, \theta^1, D_3)$ by (15.2) and (15.3). The domain of G is $X \times Y \times Z$,

$$\begin{aligned} X &= \{\bar{x}^0 : |\bar{x}^0| < \delta\}, \\ Y &= \{(h^1, \theta^1, D_3) : h^1 \in I, 0 < \theta^1 < \delta_1, 0 \leq D_3 < \delta\}, \\ Z &= \{\bar{y}^1 : |y^1| < \delta\}. \end{aligned}$$

The number δ_1 will be explained below.

The proof then goes as follows.

(1) $G(0, (h^1, \theta^1, D_3), 0) = 0$ and $G(0, (h^1, \theta^1, D_3), y^1)$ is of order $e^{-\mu\tau}$.
(2) $D_{\bar{x}^0} G(0, (h^1, \theta^1, D_3), 0) = 1$.
(3) $D_{\bar{x}^0} G(\bar{x}^0, (h^1, \theta^1, D_3), y^1) - D_{\bar{x}^0} G(0, (h^1, \theta^1, D_3), 0)$ is of order $e^{-\mu\tau}$.
(4) By the Implicit Function Theorem (version in [17]), for each $((h^1, \theta^1, D_3), y^1)$, there is a unique $\bar{x}^0$, of order $e^{-\mu\tau}$, such that $G(\bar{x}^0, (h^1, \theta^1, D_3), y^1) = 0$.
(5) Any partial derivative of G with respect to $((h^1, \theta^1, D_3), y^1)$ is of order $e^{-(\mu-2\gamma)\tau}$.
(6) Any partial derivative of $\bar{x}^0$ with respect to $((h^1, \theta^1, D_3), y^1)$ is of order $e^{-(\mu-2\gamma)\tau}$.
(7) Any partial derivative of $\tilde{x}$ with respect to $(y^1, h^1, \theta^1, D_3)$ is of order $e^{-(\mu-2\gamma)\tau}$. The result then follows by substituting (15.2) for τ.

To show (1), first note that

$$\begin{aligned} &G(0, (h^1, \theta^1, D_3), 0) = \\ &\quad \hat{x}(0, (h, r)(0, \tau, x^0, 0, h^1, \delta, \theta^1, D_3), D_3) - \hat{x}((y, h, r)(0, \tau, x^0, 0, h^1, \delta, \theta^1, D_3), D_3) = \\ &\quad\quad \hat{x}(0, (h, r)(0, \tau, x^0, 0, h^1, \delta, \theta^1, D_3), D_3) - \hat{x}(0, (h, r)(0, \tau, x^0, 0, h^1, \delta, \theta^1, D_3), D_3) = 0. \end{aligned}$$

Next note that

$$\begin{aligned} &G(0, (h^1, \theta^1, D_3), y^1) = \\ &\quad \hat{x}(0, (h_\sharp, r_\sharp)(-\tau, h^1, \delta, \theta^1, D_3), D_3) - \hat{x}((y, h, r)(0, \tau, x^0, y^1, h^1, \delta, \theta^1, D_3), D_3). \end{aligned} \tag{15.4}$$

We claim that if θ_1 is small enough (i.e., less than some δ_1), then $h_\sharp(-t, h^1, \delta, \theta^1, D_3)$ is close to h^1 for $0 \leq t \leq \tau$ and hence is in I. To see this, note that the flow of (13.48)–(13.50) with $x = 0$ or $y = 0$ can be analyzed by first dividing by θ, thus producing a system like that analyzed in the previous section.

Then by Deng's Lemma, the yhr-coordinates of the two points at which $\hat{x}$ is evaluated in (15.4) differ by $O(e^{-\mu\tau})$. Since $\hat{x}$ is differentiable, the result follows.

(2) follows from:

$$\begin{aligned} &G(\bar{x}^0, (h^1, \theta^1, D_3), 0) = \\ &\hat{x}(0, (h, r)(0, \tau, x^0, 0, h^1, \delta, \theta^1, D_3), D_3) + \bar{x}^0 - \hat{x}(0, (h, r)(0, \tau, x^0, 0, h^1, \delta, \theta^1, D_3), D_3) = \bar{x}^0. \end{aligned}$$

To show (3), write

$$D_{\bar{x}^0} G(\bar{x}^0, (h^1, \theta^1, D_3), y^1) = 1 + \frac{\partial \hat{x}}{\partial y} \frac{\partial y}{\partial x^0} \frac{\partial x^0}{\partial \bar{x}^0} + \frac{\partial \hat{x}}{\partial h} \frac{\partial h}{\partial x^0} \frac{\partial x^0}{\partial \bar{x}^0} + \frac{\partial \hat{x}}{\partial r} \frac{\partial r}{\partial x^0} \frac{\partial x^0}{\partial \bar{x}^0}.$$

The partial derivatives of $\hat{x}$ are bounded; $\frac{\partial x^0}{\partial \bar{x}^0} = 1$; and the other partial derivatives are $O(e^{-\mu\tau})$ by Deng's Lemma.

Now

$$|G(0, (h^1, \theta^1, D_3), y^1)| \leq K e^{-\mu\tau}$$

by (1), and

$$|D_{\bar{x}^0}G(\bar{x}^0,(h^1,\theta^1,D_3),y^1) - D_{\bar{x}^0}G(0,(h^1,\theta^1,D_3),0)| \le Le^{-\mu\tau}.$$

by (3). Choose δ_1 so small that for $|\theta^1| < \delta_1$, $Ke^{-\mu\tau} < \delta$ and $Le^{-\mu\tau} < \frac{1}{2}$. Then by the Implicit Function Theorem (version in [17], which also uses (2)), for each $((h^1,\theta^1,D_3),y^1) \in Y \times Z$ there is a unique $\bar{x}^0$ with $|\bar{x}^0| \le 2Ke^{-\mu\tau}$ such that $G(\bar{x}^0,(h^1,\theta^1,D_3),y^1) = 0$.

To show (5), we consider only $\frac{\partial G}{\partial \theta^1}$:

$$\frac{\partial G}{\partial \theta^1} = \frac{\partial x^0}{\partial \theta^1} - \Big(\frac{\partial \hat{x}}{\partial y}\big(\frac{\partial y}{\partial \tau}\frac{\partial \tau}{\partial \theta^1} + \frac{\partial y}{\partial x^0}\frac{\partial x^0}{\partial \theta^1} + \frac{\partial y}{\partial \theta^1}\big) + \frac{\partial \hat{x}}{\partial h}\big(\frac{\partial h}{\partial \tau}\frac{\partial \tau}{\partial \theta^1} + \frac{\partial h}{\partial x^0}\frac{\partial x^0}{\partial \theta^1} + \frac{\partial h}{\partial \theta^1}\big) + \frac{\partial \hat{x}}{\partial r}\big(\frac{\partial r}{\partial \tau}\frac{\partial \tau}{\partial \theta^1} + \frac{\partial r}{\partial x^0}\frac{\partial x^0}{\partial \theta^1} + \frac{\partial r}{\partial \theta^1}\big)\Big)$$

where

$$\frac{\partial x^0}{\partial \theta^1} = \frac{\partial \hat{x}}{\partial h}\big(-\frac{\partial h_\sharp}{\partial t}\frac{\partial \tau}{\partial \theta^1} + \frac{\partial h_\sharp}{\partial \theta^1}\big) + \frac{\partial \hat{x}}{\partial r}\big(-\frac{\partial r_\sharp}{\partial t}\frac{\partial \tau}{\partial \theta^1} + \frac{\partial r_\sharp}{\partial \theta^1}\big). \tag{15.5}$$

(For brevity we have not shown where these partial derivatives are evaluated; thus some terms that appear to be equal are not.) Since $\frac{\partial r}{\partial x^0} = 0$, one term can be ignored. From (15.2), for large τ,

$$\left|\frac{\partial \tau}{\partial \theta^1}\right| \le L\tau^2 \le Le^{\gamma\tau}.$$

Since partial derivatives of $\hat{x}$ are bounded and partial derivatives of $h_\sharp$ and $r_\sharp$ at $t = -\tau$ are of order $e^{\gamma\tau}$, $\frac{\partial x^0}{\partial \theta^1}$ is of order $e^{2\gamma\tau}$. By Deng's Lemma, partial derivatives of y at $t = 0$ are of order $e^{-\mu\tau}$, and $\frac{\partial h}{\partial x^0}$ at $t = 0$ is of order $e^{-\mu\tau}$. Therefore the following terms are of order $e^{-(\mu-2\gamma)\tau}$:

$$\frac{\partial \hat{x}}{\partial y}\big(\frac{\partial y}{\partial \tau}\frac{\partial \tau}{\partial \theta^1} + \frac{\partial y}{\partial x^0}\frac{\partial x^0}{\partial \theta^1} + \frac{\partial y}{\partial \theta^1}\big) + \frac{\partial \hat{x}}{\partial h}\frac{\partial h}{\partial x^0}\frac{\partial x^0}{\partial \theta^1}.$$

We are left with the following terms:

$$\frac{\partial \hat{x}}{\partial h}\big(-\frac{\partial h_\sharp}{\partial t}\frac{\partial \tau}{\partial \theta^1} + \frac{\partial h_\sharp}{\partial \theta^1}\big) - \frac{\partial \hat{x}}{\partial h}\big(\frac{\partial h}{\partial \tau}\frac{\partial \tau}{\partial \theta^1} + \frac{\partial h}{\partial \theta^1}\big) + \frac{\partial \hat{x}}{\partial r}\big(-\frac{\partial r_\sharp}{\partial t}\frac{\partial \tau}{\partial \theta^1} + \frac{\partial r_\sharp}{\partial \theta^1}\big) - \frac{\partial \hat{x}}{\partial r}\big(\frac{\partial r}{\partial \tau}\frac{\partial \tau}{\partial \theta^1} + \frac{\partial r}{\partial \theta^1}\big)\Big)$$

These terms may be grouped so that the terms in each group are of order $e^{-(\mu-2\gamma)\tau}$. For example, taking account where each term is evaluated:

$$\frac{\partial \hat{x}}{\partial h}\frac{\partial h_\sharp}{\partial \theta^1} - \frac{\partial \hat{x}}{\partial h}\frac{\partial h}{\partial \theta^1} = \frac{\partial \hat{x}}{\partial h}(0,(h_\sharp,r_\sharp)(-\tau,h^1,\delta,\theta^1,D_3),D_3)\frac{\partial h_\sharp}{\partial \theta^1}(-\tau,h^1,\delta,\theta^1,D_3) - \frac{\partial \hat{x}}{\partial h}((y,h,r)(0,\tau,x^0,y^1,h^1,\delta,\theta^1,D_3),D_3)\frac{\partial h}{\partial \theta^1}(0,\tau,x^0,y^1,h^1,\delta,\theta^1,D_3),D_3)$$

This is the sum of

$$\frac{\partial \hat{x}}{\partial h}(0,(h_\sharp,r_\sharp)(-\tau,h^1,\delta,\theta^1,D_3),D_3)\big(\frac{\partial h_\sharp}{\partial \theta^1}(-\tau,h^1,\delta,\theta^1,D_3) - \frac{\partial h}{\partial \theta^1}(0,\tau,x^0,y^1,h^1,\delta,\theta^1,D_3),D_3)\big)$$

and

$$\big(\frac{\partial \hat{x}}{\partial h}(0,(h_\sharp,r_\sharp)(-\tau,h^1,\delta,\theta^1,D_3),D_3) - \frac{\partial \hat{x}}{\partial h}((y,h,r)(0,\tau,x^0,y^1,h^1,\delta,\theta^1,D_3),D_3)\big) \times \frac{\partial h}{\partial \theta^1}(0,\tau,x^0,y^1,h^1,\delta,\theta^1,D_3),D_3)$$

The first summand is a bounded term times one that is $O(e^{-\mu\tau})$ by Deng's Lemma. In the second term, $\frac{\partial \hat{x}}{\partial h}$ is evaluated at two points that, by Deng's Lemma, differ by $O(e^{-\mu\tau})$. Thus the difference is $O(e^{-\mu\tau})$. This difference is multiplied by $\frac{\partial h}{\partial \theta^1}(0, \tau, x^0, y^1, h^1, \delta, \theta^1, D_3), D_3)$, which is $O(e^{\gamma\tau})$.

(6) follows from (2)–(5). To show (7), we consider only $\frac{\partial \tilde{x}}{\partial \theta^1}$. We use (15.1) to write

$$\frac{\partial \tilde{x}}{\partial \theta^1} = \frac{\partial x}{\partial t}\frac{\partial \tau}{\partial \theta^1} + \frac{\partial x}{\partial \tau}\frac{\partial \tau}{\partial \theta^1} + \frac{\partial x}{\partial x^0}\frac{\partial x^0}{\partial \theta^1} + \frac{\partial x}{\partial \theta^1},$$

with $\frac{\partial x^0}{\partial \theta^1}$ given by (15.5) plus $\frac{\partial \bar{x}^0}{\partial \theta^1}$. From these expressions and (6) we easily show (7).

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Mathematics Department, North Carolina State University, Box 8205, Raleigh, NC 27695 USA, 919-515-1876
E-mail address: vemanuki@ncsu.edu

Mathematics Department, North Carolina State University, Box 8205, Raleigh, NC 27695 USA, 919-515-6533
E-mail address: schecter@math.ncsu.edu