STABILITY OF TRAVELING WAVES FOR DEGENERATE SYSTEMS OF REACTION DIFFUSION EQUATIONS

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Abstract. For the linearization of a degenerate reaction-diffusion system at a traveling pulse or front, we prove a theorem that allows one to derive information about the semigroup generated by the linear operator from spectral information about the linear operator itself. The result is used to complete existing proofs of stability of traveling fronts for the Fitzhugh-Nagumo equation.

1. Introduction

We consider systems of the form

$$\begin{aligned} \partial_t u &= d\partial_{xx} u + \tilde{a}\partial_x u + R_1(u,v), \\ \partial_t v &= R_2(u,v), \end{aligned} \tag{1.1}$$

with

$$u = u(t,x) \in \mathbb{R}^{N_1}, \quad v = v(t,x) \in \mathbb{R}^{N_2}, \quad x \in \mathbb{R}, \quad t \geq 0,$$
$$d = \operatorname{diag}(d_1, \ldots, d_{N_1}) \text{ with all } d_k > 0, \quad \tilde{a} = (\tilde{a}_{kl}) \text{ of size } N_1 \times N_1.$$

The matrices d and $\tilde{a}$ are constant. We assume that the maps R_j are continuously differentiable. Thus (1.1) represents a system of N_1 reaction-diffusion equations and N_2 ordinary differential equations equations, coupled nonlinearly through their zeroth-order terms.

Let $w = (u,v)$, and let $w(t,x) = q(\xi)$, $\xi = x - ct$, be a traveling wave solution of (1.1) that has finite limits $q^\pm = \lim_{\xi\to\pm\infty} q(\xi)$ at infinity. We are interested in the time-asymptotic stability of $q(\xi)$. After replacing x by the moving coordinate ξ, q becomes a stationary solution.

The reaction-diffusion case, for which $N_2 = 0$, has been extensively studied [31]. In this case the linearized operator $\mathcal{L}$ at q is sectorial. Since traveling waves can be shifted, $\mathcal{L}$ always has 0 as an eigenvalue. By results of Henry [12], if (1) 0 is a simple eigenvalue of $\mathcal{L}$, (2) the rest of the spectrum of $\mathcal{L}$ is contained in $\mathrm{Re}\lambda < -\nu < 0$, and (3) the nonlinear term is small, then the traveling wave is asymptotically stable.

The degenerate reaction-diffusion case, for which N_1 and N_2 are both positive, includes the FitzHugh–Nagumo equation and other models for electrical activity in neurons; certain combustion models [9]; population models in which some species diffuse and others do not

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[11], [21, Sec. 13.8], [22, pp. 7–9], [22, Secs. 13.4, 13.5, 13.9]; and "buffered" reaction-diffusion systems, in which a diffusing reaction product can be absorbed by stationary buffers [33, 15].

In this case $\mathcal{L}$ is not sectorial; it generates a C_0-semigroup, but not an analytic semigroup. According to a well-known result of Bates and Jones [1], if (1) $e^{\mathcal{L}}$ has a simple eigenvalue 1, (2) the rest of the spectrum of $e^{\mathcal{L}}$ is contained in $|\lambda| < \rho < 1$, and (3) the nonlinear term is small, then the traveling wave is asymptotically stable. When $\mathcal{L}$ is sectorial, the spectrum of $e^{\mathcal{L}}$ is obtained by exponentiating the spectrum of $\mathcal{L}$; when $\mathcal{L}$ is not sectorial, however, this need not be the case [7, 27, 34].

If $q^- = q^+$, the traveling wave is a pulse. For pulses, an argument of Evans [8], later simplified by Bates and Jones [1], shows, roughly speaking, that the spectrum of $e^{\mathcal{L}}$ is obtained by exponentiating the spectrum of $\mathcal{L}$. This result is key to the proof of stability of traveling pulses in the FitzHugh–Nagumo equation. The arguments of these authors use the fact that $e^{\mathcal{L}}$ is a compact perturbation of $e^{\mathcal{L}_0}$, where $\mathcal{L}_0$ is the linearization of the partial differential operator at $q^- = q^+$, which is a constant-coefficient operator.

If $q^- \neq q^+$, the traveling wave is a front. Fronts in the FitzHugh–Nagumo equation have been shown to exist by Yanagida [35] (simple fronts) and Deng [6] (more complicated fronts), and the spectrum of the linearized operator at these solutions has been carefully studied by Yanagida [35] (simple fronts) and by Nii [23] and Sandstede [30] (more complicated fronts). However, since the linearizations at q^- and q^+ are different, arguments similar to those of Evans or Bates and Jones cannot be used to derive asymptotic stability of the traveling wave from information about the spectrum of $\mathcal{L}$.

In some situations, if a traveling wave is perturbed in one norm, then the solution converges to the traveling wave in a second norm. In these situations, the result of Bates and Jones cannot be used. However, the linear portion of the analysis, in which one tries to obtain knowledge about the spectrum of $e^{\mathcal{L}}$ from knowledge about the spectrum of $\mathcal{L}$, should be similar to what is required in more standard stability proofs.

In [9] we studied stability of a front in a combustion problem of degenerate reaction-diffusion type. The passage from information about the spectrum of $\mathcal{L}$ to information about the spectrum of $e^{\mathcal{L}}$ was nonstandard, but was made easier by the fact that $\mathcal{L}$ was a 2×2 matrix of differential operators for which one off-diagonal operator was compact. For the linearized operator at a FitzHugh–Nagumo front, this is not the case, so the techniques of [9] do not allow one to complete the stability proofs in [35], [23], or [30].

In this paper we prove a general theorem that allows one, in studying the stability of a pulse or front in a degenerate reaction-diffusion problem, to pass from knowledge about the spectrum of $\mathcal{L}$ to knowledge about the spectrum of $e^{\mathcal{L}}$. No compactness assumption is required. In particular, our theorem implies that if 0 is in the discrete spectrum of $\mathcal{L}$, with a generalized eigenspace of dimension k, and is the only point of the spectrum that is located in the closed right half-plane, then the corresponding semigroup is uniformly exponentially stable on a subspace of codimension k. For pulses, this result is not new, but for fronts, it is. We use our result to complete the stability proofs in [35], [23], and [30]. A feature of our approach is the use of the Greiner Spectral Mapping Theorem 6.1, which is a Banach space replacement for the frequently used Gearhart-Prüss Spectral Mapping Theorem for C_0-semigroups on Hilbert spaces.

In Section 2 we discuss the system (1.1) in more detail, show that some relevant operators generate semigroups, and recall various spectra and bounds associated with semigroups and their generators. In Section 3 we state our theorem precisely and sketch the proof. Sections 4–9 and Appendix A are devoted to the proof. The application to Fitzhugh–Nagumo fronts is given in Section 10. Concluding discussion is in Section 11.

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Note added May 29, 2011. In earlier versions of this paper, the second equation of (1.1) was $\partial_t v = \tilde{b}\partial_t v + R_2(u,v)$, with $\tilde{b}$ a real diagonal matrix whose entries are not necessarily equal. In this case the number c in (2.1) would be replaced by a diagonal matrix. However, the proof we give is not valid in this more general case; see Remark 5.7.

After the original version of this paper was submitted, the Ph.D. thesis of Jens Rottmann-Matthes [28], which deals with a more general class of equations than (1.1) by completely different methods, appeared. A paper on linearized stability of hyperbolic problems, based on part of this thesis, has recently been published [29].

2. System and definitions

2.1. **System to be studied.** Let us assume that in (1.1), N_1 and N_2 are both positive. After replacing x by $\xi = x - ct$, (1.1) becomes

$$\begin{aligned} \partial_t u &= d\partial_{\xi\xi} u + a\partial_\xi u + R_1(u,v), \\ \partial_t v &= c\partial_\xi v + R_2(u,v), \end{aligned} \tag{2.1}$$

with $a = \tilde{a} + \mathrm{diag}(c,\dots,c)$.

We denote the differentials of the maps R_j by $R_{j1} = \partial_u R_j$, $R_{j2} = \partial_v R_j$. Therefore the linearization of (2.1) at $q(\xi) = (q_1(\xi), q_2(\xi))$, with $c = 1$, is

$$\begin{aligned} \partial_t u &= d\partial_{\xi\xi} u + a\partial_\xi u + R_{11}(q_1(\xi), q_2(\xi))u + R_{12}(q_1(\xi), q_2(\xi))v, \\ \partial_t v &= c\partial_\xi v + R_{21}(q_1(\xi), q_2(\xi))u + R_{22}(q_1(\xi), q_2(\xi))v. \end{aligned} \tag{2.2}$$

To simplify the exposition, we shall always assume, without explicitly recalling it:

(H) $c \neq 0$.

However, this assumption is not necessary; see Section 11.

Let $B_{ij}(\xi) = R_{ij}(q_1(\xi), q_2(\xi))$. Associated with (2.2) is the linear differential operator

$$\mathcal{L} = \begin{pmatrix} d\partial_{\xi\xi} + a\partial_\xi & 0 \\ 0 & c\partial_\xi \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} \mathcal{A} & B_{12} \\ B_{21} & \mathcal{G} \end{pmatrix}, \quad \xi \in \mathbb{R}, \tag{2.3}$$

where $\mathcal{A}$ and $\mathcal{G}$ are differential operators given by

$$\mathcal{A} = d\partial_{\xi\xi} + a\partial_\xi + B_{11}, \quad \mathcal{G} = c\partial_\xi + B_{22}. \tag{2.4}$$

Note that we use the same notation for a (matrix-valued) function and the operator of multiplication by that function.

We assume that the traveling wave $q(\xi)$ approaches its ends states exponentially. Then the matrix valued function $B = (B_{ij}) : \mathbb{R} \to \mathbb{R}^{(N_1+N_2)^2}$ satisfies the following conditions. In our statements of results, we will indicate which of these assumptions are actually being used.

Hypothesis 2.1. (a) *B is bounded and continuous, and there are matrices $B^\pm$ such that $B(\xi) \to B^\pm$ exponentially as $\xi \to \pm\infty$.*

(b) *B is differentiable, with bounded and continuous derivative B', and $B'(\xi) \to 0$ exponentially as $\xi \to \pm\infty$.*

Remark 2.2. The exponential convergence in condition (a) is used to ensure that if B^- (respectively B^+) has an exponential dichotomy on $\mathbb{R}_-$ (respectively $\mathbb{R}_+$), then so does $B(\xi)$. It could be replaced by any other condition that leads to the same conclusion, for example by the condition $\|B(\xi) - B^{\pm}\| \in L^1(\mathbb{R}_{\pm})$ [4, 5].

Remark 2.3. All results hold if the matrix a is complex-valued and the mapping B takes its values in the space of complex-vaued $(N_1 + N_2) \times (N_1 + N_2)$-matrices, provided Hypothesis 2.1 holds.

2.2. **Semigroups.** If the matrix-valued function B is replaced by the constant matrix $B^{\pm}$, we obtain constant-coefficient differential operators analogous to $\mathcal{L}$, $\mathcal{A}$, and $\mathcal{G}$ that we denote $\mathcal{L}^{\pm}$, $\mathcal{A}^{\pm}$, and $\mathcal{G}^{\pm}$ respectively.

Let $\mathcal{E}_0$ denote one of the spaces $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$; the latter is the space of bounded uniformly continuous functions with the sup norm. The operators $\mathcal{A}$ and $\mathcal{A}^{\pm}$ (respectively $\mathcal{G}$ and $\mathcal{G}^{\pm}$, $\mathcal{L}$ and $\mathcal{L}^{\pm}$) will be considered on the space $\mathcal{E}_0^{N_1}$ (respectively $\mathcal{E}_0^{N_2}$, $\mathcal{E}_0^{N_1} \oplus \mathcal{E}_0^{N_2}$). The domain of $\mathcal{A}$ and $\mathcal{A}^{\pm}$ is the direct sum of N_1 copies of the natural domain of $\partial_{\xi\xi}$ on $\mathcal{E}_0$; the domain of $\mathcal{G}$ and $\mathcal{G}^{\pm}$ is the direct sum of N_2 copies of the natural domain of ∂_ξ on $\mathcal{E}_0$; and the domain of $\mathcal{L}$ and $\mathcal{L}^{\pm}$ is the direct sum of the domains of $\mathcal{A}$ and $\mathcal{G}$. (The natural domain of, for example, ∂_ξ on $\mathcal{E}_0 = L^2(\mathbb{R})$, is $H^1(\mathbb{R})$.)

The operator $d\partial_{\xi\xi} + a\partial_\xi$ on $\mathcal{E}_0^{N_1}$ is sectorial (see [12], pp. 136–137, and [26], Section 3.2, Corollary 2.3) and hence generates an analytic semigroup ([12], Theorem 1.3.4). The operator $c\partial_\xi$ on $\mathcal{E}_0^{N_2}$ generates the semigroup $P(t)v(\xi) = v(\xi + ct)$, which is clearly a C_0-semigroup. Therefore the diagonal operator

$$\begin{pmatrix} d\partial_{\xi\xi} + a\partial_\xi & 0 \\ 0 & c\partial_\xi \end{pmatrix} \tag{2.5}$$

generates a C_0-semigroup on $\mathcal{E}_0^{N_1} \times \mathcal{E}_0^{N_2}$. If $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, we assume Hypothesis 2.1 (a), which implies that the multiplication operator B is bounded on $\mathcal{E}_0$; if $\mathcal{E}_0 = H^1(\mathbb{R})$, we assume Hypothesis 2.1 (b), which again implies that B is bounded on $\mathcal{E}_0$. Then the operators $\mathcal{A}$ and $\mathcal{A}^{\pm}$, $\mathcal{G}$ and $\mathcal{G}^{\pm}$, and $\mathcal{L}$ and $\mathcal{L}^{\pm}$ are bounded perturbations of, respectively, $d\partial_{\xi\xi} + a\partial_\xi$, $c\partial_\xi$, and the diagonal operator (2.5). Therefore $\mathcal{A}$ and $\mathcal{A}^{\pm}$ generate analytic semigroups ([26], Section 3.2, Corollary 2.2), and each of the other operators generates a C_0-semigroup ([26], Section 3.1, Theorem 1.1).

2.3. **Spectra and bounds.** Let $\mathcal{X}$ be a Banach space, and let $\mathcal{C} : \mathcal{X} \to \mathcal{X}$ be a closed, densely defined linear operator. Its *resolvent set* $\rho(\mathcal{C})$ is the set of $\lambda \in \mathbb{C}$ such that $\mathcal{C} - \lambda I$ has a bounded inverse. The complement of $\rho(\mathcal{C})$ is the *spectrum* $\mathrm{Sp}(\mathcal{C})$. It is the union of the *discrete spectrum* $\mathrm{Sp}_{\mathrm{d}}(\mathcal{C})$, which is the set of isolated eigenvalues of $\mathcal{C}$ of finite algebraic multiplicity, and the *essential spectrum* $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{C})$, which is the rest. We also define the *point spectrum* $\mathrm{Sp}_p(\mathcal{C})$, which is the set of all eigenvalues of $\mathcal{C}$.

$\mathcal{C}$ is *Fredholm* if its range is closed, its kernel has finite dimension n, and its range has finite codimension m. The *index* of a Fredholm operator $\mathcal{C}$ is $n - m$. The *Fredholm resolvent set* $\rho_{\mathrm{F}}(\mathcal{C})$ is the set of all $\lambda \in \mathbb{C}$ such that $\mathcal{C} - \lambda I$ is Fredholm of index zero. The set $\rho_{\mathrm{F}}(\mathcal{C})$ is open, and its complement, the *Fredholm spectrum* $\mathrm{Sp}_{\mathrm{F}}(\mathcal{C})$, is contained in $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{C})$.

We define:

- The *spectral bound* $\mathrm{s}(\mathcal{C}) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}(\mathcal{C})\}$.
- The *essential spectral bound* $\mathrm{s}_{\mathrm{ess}}(\mathcal{C})$, the infimum of all real ω such that $\mathrm{Sp}(\mathcal{C}) \cap \{\lambda : \mathrm{Re}\,\lambda > \omega\}$ is a subset of $\mathrm{Sp}_{\mathrm{d}}(\mathcal{C})$ and has only finitely many points.

- The *Fredholm spectral bound* $\mathrm{s_F}(\mathcal{C}) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp_F}(\mathcal{C})\}$.

For a *bounded* linear operator $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$, we define:

- The *spectral radius* of $\mathcal{T}$, the supremum of $\{|\lambda| : \lambda \in \mathrm{Sp}(\mathcal{T})\}$.
- The *essential spectral radius* of $\mathcal{T}$, the supremum of $\{|\lambda| : \lambda \in \mathrm{Sp_{ess}}(\mathcal{T})\}$.
- The seminorm
$$\|\mathcal{T}\|_C = \inf_{\mathcal{K}} \|\mathcal{T} + \mathcal{K}\|,$$
where the infimum is over the set of all compact operators $\mathcal{K} : \mathcal{X} \to \mathcal{X}$.

If $\mathcal{C}$ generates a C_0-semigroup $\mathcal{T}(t)$, $t \geq 0$, we define:

- The *growth bound* $\omega(\mathcal{C}) = \lim_{t\to\infty} t^{-1} \log \|\mathcal{T}(t)\|$.
- The *essential growth bound* $\omega_{\mathrm{ess}}(\mathcal{C}) = \lim_{t\to\infty} t^{-1} \log \|\mathcal{T}(t)\|_C$.

The following proposition summarizes various well-known facts about these sets and numbers.

Proposition 2.4. *Suppose $\mathcal{C} : \mathcal{X} \to \mathcal{X}$ generates the C_0-semigroup $e^{t\mathcal{C}}$, $t \geq 0$. Then*

(1) *There exists $\omega \in \mathbb{C}$ such that $\{\lambda : \mathrm{Re}\,\lambda > \omega\} \subset \rho(\mathcal{C})$.*
(2) *For each $t > 0$, $e^{t\,\mathrm{Sp}(\mathcal{C})} \subset \mathrm{Sp}(e^{t\mathcal{C}})$, $e^{t\,\mathrm{Sp_{ess}}(\mathcal{C})} \subset \mathrm{Sp_{ess}}(e^{t\mathcal{C}})$, and $e^{t\,\mathrm{Sp}_p(\mathcal{C})} = \mathrm{Sp}_p(e^{t\mathcal{C}}) \setminus \{0\}$.*
(3) $\mathrm{s_F}(\mathcal{C}) \leq \mathrm{s_{ess}}(\mathcal{C}) \leq \mathrm{s}(\mathcal{C})$.
(4) $\mathrm{s}(\mathcal{C}) \leq \omega(\mathcal{C})$ *and* $\mathrm{s_{ess}}(\mathcal{C}) \leq \omega_{\mathrm{ess}}(\mathcal{C})$.
(5) *For each $t > 0$, $e^{t\omega(\mathcal{C})}$ is the spectral radius of $e^{t\mathcal{C}}$, and $e^{t\omega_{\mathrm{ess}}(\mathcal{C})}$ is the essential spectral radius of $e^{t\mathcal{C}}$.*
(6) *Let $\omega > \omega_{\mathrm{ess}}(\mathcal{C})$ be a number such that no element of $\mathrm{Sp}(\mathcal{C})$ has real part ω. Then there is a finite set $\{\lambda_1, \dots, \lambda_k\} \subset \mathbb{C}$ such that*
$$\mathrm{Sp}(\mathcal{C}) \cap \{\lambda : \mathrm{Re}\,\lambda > \omega\} = \mathrm{Sp_d}(\mathcal{C}) \cap \{\lambda : \mathrm{Re}\,\lambda > \omega\} = \{\lambda_1, \dots, \lambda_k\}.$$
Let $E_1, \dots, E_k$ be the generalized eigenspaces of $\lambda_1, \dots, \lambda_k$ respectively; they are finite-dimensional. Then there is a closed subspace E_0 of $\mathcal{X}$ such that $\mathcal{X} = E_0 \times E_1 \times \cdots \times E_k$ and E_0 is invariant under $\mathcal{C}$. Moreover, there is a number $K > 0$ such that $\|e^{t\mathcal{C}}|E_0\| \leq Ke^{\omega t}$.

Proof. For the equality $e^{t\,\mathrm{Sp}_p(\mathcal{C})} = \mathrm{Sp}_p(e^{t\mathcal{C}})\setminus\{0\}$, see [7, Sec. IV.3b]. The rest of the proposition is discussed in [9] where further references are provided. □

3. Results and sketch of proof

Our main result is the following theorem:

Theorem 3.1. *Assume Hypothesis 2.1 (a) if $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, and Hypothesis 2.1 (b) if $\mathcal{E}_0 = H^1(\mathbb{R})$. Then $\mathrm{s_F}(\mathcal{L}) = \omega_{\mathrm{ess}}(\mathcal{L})$.*

The point of this result is that $\mathrm{s_F}(\mathcal{L})$ can in principle be calculated, as we shall see in Section 5. The theorem then gives important information about the semigroup $e^{t\mathcal{L}}$. This information can be used, typically together with the theorem of Bates and Jones [1], to give information about the stability of the traveling wave.

In Section 5 we shall discuss the operators $\mathcal{A}$, $\mathcal{A}^{\pm}$, $\mathcal{G}$, and $\mathcal{G}^{\pm}$, for which the analog of Theorem 3.1 is known. $\mathcal{A}$ and $\mathcal{A}^{\pm}$ are sectorial, so each has a spectrum that is contained in a sector of the complex plane and generates an analytic semigroup; $\mathcal{G}$ and $\mathcal{G}^{\pm}$ are related to operators that generate evolutionary semigroups [3], so each has a spectrum that is a union of vertical lines. The difficulty with $\mathcal{L}$ is that it mixes these two distinct types of operators.

This difficulty can be seen even at the level of constant-coefficient operators. As we shall see in Section 5, for $\mathcal{L}^{\pm}$, the analog of Theorem 3.1 for $\mathcal{E}_0 = L^2(\mathbb{R})$ is easy, but for $\mathcal{E}_0 = BUC(\mathbb{R})$, as far as we know, it is not.

In outline, the proof of Theorem 3.1 goes as follows. From Proposition 2.4 (3) and (4), $\mathrm{s}_{\mathrm{F}}(\mathcal{L}) \le \omega_{\mathrm{ess}}(\mathcal{L})$. Then we only need to show that

$$\omega_{\mathrm{ess}}(\mathcal{L}) \le \mathrm{s}_{\mathrm{F}}(\mathcal{L}). \tag{3.1}$$

As in [9], Section 4.2, if Hypothesis 2.1 (b) holds, then (3.1) for $\mathcal{E}_0 = H^1(\mathbb{R})$ follows from the corresponding inequality for $\mathcal{E}_0 = L^2(\mathbb{R})$. (This is proved by the following compactness argument. For $\mathcal{E}_0 = L^2(\mathbb{R})$, the operator of differentiation $(\partial_\xi)_{L^2}$ has domain $H^1(\mathbb{R})$ and spectrum $i\mathbb{R}$. Hence the operator $\mathcal{D} = (\partial_\xi)_{L^2} + I$ is an isomorphism of $H^1(\mathbb{R})$ onto $L^2(\mathbb{R})$. We note that $\mathcal{D}B_{H^1} = B_{L^2}\mathcal{D} + B'_{L^2}\mathcal{J}$, where $\mathcal{J}$ is the imbedding of $H^1(\mathbb{R})$ in $L^2(\mathbb{R})$. Due to Hypothesis 2.1 (b), the operator B'_{L^2} of multiplication by $B'(\xi)$ is easily seen to be compact on $L^2(\mathbb{R})$. It follows that $\mathcal{D}\mathcal{L}_{H^1}\mathcal{D}^{-1} = \mathcal{L}_{L^2} + \mathcal{K}$, where $\mathcal{K}$ is a compact operator. Therefore $\omega_{\mathrm{ess}}(\mathcal{L}_{L^2}) = \omega_{\mathrm{ess}}(\mathcal{L}_{H^1})$ and $\mathrm{s}_{\mathrm{F}}(\mathcal{L}_{L^2}) = \mathrm{s}_{\mathrm{F}}(\mathcal{L}_{H^1})$; the result follows.) Hence we need only consider the cases $\mathcal{E}_0 = L^2(\mathbb{R})$ and $\mathcal{E}_0 = BUC(\mathbb{R})$. For these cases we shall only need Hypothesis 2.1 (a).

In Section 4 we shall show

Proposition 3.2. *Assume that $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, and that Hypothesis 2.1 (a) holds. Then $\mathrm{s}_{\mathrm{F}}(\mathcal{L}) = \max\{\mathrm{s}_{\mathrm{F}}(\mathcal{L}^-), \mathrm{s}_{\mathrm{F}}(\mathcal{L}^+)\}$*

This proposition is essentially known. For the sectorial operator $\mathcal{A}$, which is the type of operator that arises when one linearizes a reaction-diffusion problem at a pulse or front, a theorem of Sandstede and Scheel [32] relates Fredholm properties of $\mathcal{A}$ to Fredholm properties of an associated first-order nonautonomous linear ODE on $-\infty < \xi < \infty$. Then Palmer's Theorem (references in Section 4) relates Fredholm properties of this linear ODE to the constant-coefficient linear ODEs obtained by setting $\xi = \pm\infty$. The argument of Sandstede and Scheel easily generalizes to operators like $\mathcal{L}$ that arise for degenerate reaction-diffusion problems. We give the generalization of Sandstede and Scheel's argument in Appendix A, and the rest of the proof of Proposition 3.2 in Section 4.

In Sections 6–9 we shall show

Proposition 3.3. *Assume that $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, and that Hypothesis 2.1 (a) holds. Then $\omega_{\mathrm{ess}}(\mathcal{L}) \le \max\{\mathrm{s}_{\mathrm{F}}(\mathcal{L}^-), \mathrm{s}_{\mathrm{F}}(\mathcal{L}^+)\}$.*

The inequality (3.1), and hence Theorem 3.1, follows from Propositions 3.2 and 3.3.

The proof of Proposition 3.3 is the main contribution of this paper. We make heavy use of the facts that $\mathcal{A}$ and $\mathcal{A}^{\pm}$ each has a spectrum that is contained in a sector of the complex plane, while $\mathcal{G}$ and $\mathcal{G}^{\pm}$ each has a spectrum that is a union of vertical lines; see Section 5. A key step is to show that if $\operatorname{Re} z > \mathrm{s}_{\mathrm{F}}(\mathcal{G})$ and $z + 2\pi ik \in \rho(\mathcal{L})$ for all $k \in \mathbb{Z}$, then $e^z \in \rho(e^{\mathcal{L}})$. We first show (Lemma 7.2) that $\sup_{|k|\ge K} \|(\mathcal{L} - (z + 2\pi ik)I)^{-1}\| < \infty$ for some $K \ge 0$. For $\mathcal{E}_0$ a Hilbert space, the result then follows from the Gearhart-Prüss Spectral Mapping Theorem (see, e.g., [34, Theorem 2.2.4]). For $\mathcal{E}_0 = BUC(\mathbb{R})$, however, the Gearhart-Prüss Theorem does not apply. Instead we base the proof on the Greiner Spectral Mapping Theorem (see, e.g., [34]), which we review in Section 6. Thus we must show that for each $w \in BUC(\mathbb{R})^{N_1+N_2}$, $\sum_{|k|\ge K}(\mathcal{L} - (z + 2\pi ik)I)^{-1}w$ is Cesaro summable. This is the content of Lemma 7.4, which is proved in Section 9. Another key step is to show that if a finite number of the $z + 2\pi ik$ are in $\mathrm{Sp}_\mathrm{d}(\mathcal{L})$, then $e^z \in \mathrm{Sp}_\mathrm{d}(e^{\mathcal{L}})$. This fact is the content of Lemma 6.2, a consequence of Greiner's Theorem, which we prove in Section 6.

Definition 3.4. Let $\mathcal{X}$ be a Banach space, and let $\mathcal{C} : \mathcal{X} \to \mathcal{X}$ be a closed, densely defined linear operator that generates a C_0-semigroup $e^{t\mathcal{C}}$. In this paper we shall say that $\mathcal{C}$ is *spectrally stable* if

(1) 0 is a simple eigenvalue of $\mathcal{C}$, and
(2) the rest of the spectrum of $\mathcal{C}$ lies in $\mathrm{Re}\,\lambda < -\nu$ for some $\nu > 0$.

We shall say that the semigroup $e^{t\mathcal{C}}$ is *linearly stable* if

(1) $e^{t\mathcal{C}}$ has a simple eigenvalue 1, and
(2) $e^{t\mathcal{C}}$ has a codimension-one invariant subspace $\mathcal{W}$ such $\|e^{t\mathcal{C}}|\mathcal{W}\| \le Ke^{-\nu t}$ for some $K > 0$ and $\nu > 0$.

Theorem 3.1 has the following corollary.

Corollary 3.5. *Assume Hypothesis 2.1 (a) if $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, and Hypothesis 2.1 (b) if $\mathcal{E}_0 = H^1(\mathbb{R})$. Suppose (1) $\mathrm{s_F}(\mathcal{L}) < 0$ and (2) the only element of $\mathrm{Sp}(\mathcal{L})$ in $\mathrm{Re}\,\lambda \ge 0$ is the simple eigenvalue 0. Then $\mathcal{L}$ is spectrally stable, and $e^{t\mathcal{L}}$ is linearly stable.*

Proof. By Theorem 3.1, $\omega_{\mathrm{ess}}(\mathcal{L}) < 0$. Therefore by Proposition 2.4 (6) we can choose $\nu > 0$ such that $\omega_{\mathrm{ess}}(\mathcal{L}) < -\nu$ and the only element of $\mathrm{Sp}(\mathcal{L})$ with real part greater than or equal to $-\nu$ is 0. Since the eigenvalue 0 is simple, $\mathcal{L}$ is spectrally stable. Using Proposition 2.4 (6) again, we see that $\mathcal{E}_0^{N_1+N_2}$ can be decomposed as the direct sum of two closed subspaces invariant under $\mathcal{L}$: the first, $\mathcal{W}$, has codimension 1, and the second, with dimension 1, is the eigenspace of $\mathcal{L}$ for the eigenvalue 0. Furthermore, there exists $K > 0$ such that $\|e^{t\mathcal{L}}|\mathcal{W}\| \le Ke^{-\nu t}$. Thus $e^{t\mathcal{L}}$ is linearly stable. □

Essentially the same argument proves the following slightly more general corollary of Theorem 3.1, which was mentioned in the introduction.

Corollary 3.6. *Assume Hypothesis 2.1 (a) if $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, and Hypothesis 2.1 (b) if $\mathcal{E}_0 = H^1(\mathbb{R})$. Suppose (1) $\mathrm{s_F}(\mathcal{L}) < 0$; (2) the only element of $\mathrm{Sp}(\mathcal{L})$ in $\mathrm{Re}\,\lambda \ge 0$ is 0; (3) $0 \in \mathrm{Sp_d}(\mathcal{L})$; and (4) the generalized eigenspace for the eigenvalue 0 has dimension k. Then there are numbers $K > 0$ and $\nu > 0$, and a closed subspace W of $\mathcal{E}_0^{N_1+N_2}$ of codimension k, invariant under $\mathcal{L}$, such that $\|e^{t\mathcal{L}}|W\| \le Ke^{-\nu t}$.*

4. Proof of Proposition 3.2 via Palmer's Theorem

Let (u, v) denote an element of $\mathcal{E}_0^{N_1} \times \mathcal{E}_0^{N_2}$, where $\mathcal{E}_0$ is either $L^2(\mathbb{R})$ or $BUC(\mathbb{R})$. Consider the $(2N_1 + N_2) \times (2N_1 + N_2)$ matrix-valued function

$$\mathbb{A}_z(\xi) = \begin{pmatrix} 0 & I & 0 \\ -d^{-1}(B_{11}(\xi) - zI) & -d^{-1}a & -d^{-1}B_{12}(\xi) \\ -c^{-1}B_{21}(\xi) & 0 & -c^{-1}(B_{22}(\xi) - zI) \end{pmatrix}, \tag{4.1}$$

and the related first-order differential operator $\mathcal{T}_z = \partial_\xi - \mathbb{A}_z(\xi)$ on $\mathcal{E}_0^{N_1} \times \mathcal{E}_0^{N_1} \times \mathcal{E}_0^{N_2}$. The domain of $\mathcal{T}_z$ is the direct sum of the natural domains of ∂_ξ on $\mathcal{E}_0$. The eigenvalue problem $(\mathcal{L} - zI)(u, v) = 0$ is formally equivalent to the equation $\mathcal{T}_z(u, u', v) = 0$. In fact we have:

Theorem 4.1. *The operator $\mathcal{L} - zI$ is Fredholm on $\mathcal{E}_0^{N_1+N_2}$ if and only if the operator $\mathcal{T}_z$ is Fredholm on $\mathcal{E}_0^{2N_1+N_2}$. In this case $\dim\ker(\mathcal{L} - zI) = \dim\ker\mathcal{T}_z$, $\mathrm{codim}\,\mathrm{rg}(\mathcal{L} - zI) = \mathrm{codim}\,\mathrm{rg}\,\mathcal{T}_z$, and hence $\mathrm{ind}(\mathcal{L} - zI) = \mathrm{ind}\,\mathcal{T}_z$.*

This result is the generalization of Sandstede and Scheel's result in [32] for the case $N_2 = 0$ that was mentioned in the previous section. The proof is in Appendix A.

Palmer's Theorem, which we now state, was proved in [24, 25] for $\mathcal{E}_0 = BUC(\mathbb{R})$, and in [2] for $\mathcal{E}_0 = L^2(\mathbb{R})$; see [16, 17, 20, 31, 32] for more recent discussions.

Theorem 4.2. *Let $\xi \mapsto \mathbb{A}(\xi)$ is a bounded continuous $(N \times N)$-matrix-valued function on $\mathbb{R}$, and let $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$. Then the operator $\mathcal{T} = \partial_\xi - \mathbb{A}(\xi)$ on $\mathcal{E}_0^N$ is Fredholm if and only if the differential equation $\partial_\xi u = \mathbb{A}(\xi)u$ has exponential dichotomies with projections $P^\pm(\xi)$ on $\mathbb{R}_\pm$. In this case* $\dim\ker\mathcal{T} = \dim(\operatorname{rg} P^+(0) \cap \ker P^-(0))$, $\operatorname{codim}\operatorname{rg}\mathcal{T} = \operatorname{codim}(\operatorname{rg} P^+(0) + \ker P^-(0))$, *and* $\operatorname{ind}\mathcal{T} = \dim\operatorname{rg} P^+(0) - \dim\operatorname{rg} P^-(0)$.

Remark 4.3. Palmer's Theorem implies that the operator $\mathcal{T}$ on $L^2(\mathbb{R})^N$ is Fredholm if and only if the corresponding operator on $BUC(\mathbb{R})^N$ is Fredholm, and these operators have kernels and cokernels of the same dimension. Actually, on either space, $\ker\mathcal{T}$ is the space of solutions of $\partial_\xi u = \mathbb{A}(\xi)u$ with $u(0) \in \operatorname{rg} P^+(0) \cap \ker P^-(0)$.

The following lemma gathers some facts that we will need. Statement (2) follows from standard results on persistence of exponential dichotomies [4, 5].

Lemma 4.4. *Assume Hypothesis 2.1 (a) and (b). Then:*

1. *There are matrices $\mathbb{A}_z(\pm\infty)$ such that $\mathbb{A}_z(\xi) \to \mathbb{A}_z(\pm\infty)$ exponentially as $\xi \to \pm\infty$.*
2. *The differential equation $\partial_\xi u = \mathbb{A}_z(\xi)u$ has an exponential dichotomy on $\mathbb{R}_\pm$ if and only if the differential equation $\partial_\xi u = \mathbb{A}_z(\pm\infty)u$ has an exponential dichotomy on $\mathbb{R}_\pm$; in this case the dichotomy projections for the two equations have equal ranks.*
3. *A constant-coefficient linear differential equation $\partial_\xi u = \mathbb{A}u$ has an exponential dichotomy on $\mathbb{R}_-$ or $\mathbb{R}_+$ if and only if* $\operatorname{Sp}(\mathbb{A}) \cap i\mathbb{R} = \emptyset$. *On $\mathbb{R}_-$, the associated projection has kernel equal to the sum of the generalized eigenspaces for the eigenvalues with positive real part. On $\mathbb{R}_+$, the associated projection has range equal to the sum of the generalized eigenspaces for the eigenvalues with negative real part.*

Theorems 4.1 and 4.2, together with Lemma 4.4, yield:

Corollary 4.5. *Let $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$. The following are equivalent.*

1. *The operator $\mathcal{L} - zI$ is Fredholm of index k.*
2. $\operatorname{Sp}(\mathbb{A}_z(-\infty)) \cap i\mathbb{R} = \emptyset$, $\operatorname{Sp}(\mathbb{A}_z(\infty)) \cap i\mathbb{R} = \emptyset$, *and*

$$\text{number of eigenvalues of } \mathbb{A}_z(-\infty) \text{ with negative real part} \\ - \text{number of eigenvalues of } \mathbb{A}_z(\infty) \text{ with negative real part} = k.$$

Remark 4.6. Let $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$. From Theorems 4.1 and 4.2, the following are equivalent:

1. The operator $\mathcal{L} - zI$ is invertible.
2. The operator $\mathcal{T}_z$ is invertible.
3. The differential equation $\partial_\xi u = \mathbb{A}_z(\xi)u$ has an exponential dichotomy on $\mathbb{R}$.
4. The differential equation $\partial_\xi u = \mathbb{A}_z(\xi)u$ has exponential dichotomies on $\mathbb{R}_-$ and $\mathbb{R}_+$ such that $P^-(0) = P^+(0)$.

Because of Lemma 4.4, Theorem 4.2 has the following corollary.

Corollary 4.7. *Let $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$. Let $\mathbb{A}$ be a constant matrix, and let $\mathcal{T} = \partial_\xi - \mathbb{A}$. Then the following are equivalent:*

(1) $\mathcal{T}$ *is invertible.*
(2) $\mathcal{T}$ *Fredholm.*
(3) *The differential equation* $\partial_\xi u = \mathbb{A}u$ *has an exponential dichotomy on* $\mathbb{R}$.
(4) $\mathrm{Sp}(\mathbb{A}) \cap i\mathbb{R} = \emptyset$.

Combining Theorem 4.1, with $\mathcal{L}$ replaced by $\mathcal{L}^{\pm}$, and Corollary 4.7, we have:

Corollary 4.8. *Let* $\mathcal{E}_0 = L^2(\mathbb{R})$ *or* $BUC(\mathbb{R})$. *The following are equivalent:*
(1) $z \in \rho(\mathcal{L}^-)$.
(2) $\mathcal{L}^- - zI$ *is Fredholm.*
(3) $\mathrm{Sp}(\mathbb{A}_z(-\infty)) \cap i\mathbb{R} = \emptyset$.
An analogous result holds for $\mathcal{L}^+$*; in (3) replace* $\mathbb{A}_z(-\infty)$ *by* $\mathbb{A}_z(\infty)$.

If S is an open subset of $\mathbb{C}$ such that for some ω, $\{\lambda : \mathrm{Re}\,\lambda > \omega\} \subset S$, then we denote by S^∞ the component of S that contains $\{\lambda : \mathrm{Re}\,\lambda > \omega\}$.

Proposition 4.9. *Assume that* $\mathcal{E}_0 = L^2(\mathbb{R})$ *or* $BUC(\mathbb{R})$, *and that Hypothesis 2.1 (a) holds. Then* $\rho_{\mathrm{F}}(\mathcal{L})^\infty = (\rho_{\mathrm{F}}(\mathcal{L}^-) \cap \rho_{\mathrm{F}}(\mathcal{L}^+))^\infty$.

Proof. Let $z \in \rho_{\mathrm{F}}(\mathcal{L})^\infty$. Let $z(\sigma)$ be a path in $\rho_{\mathrm{F}}(\mathcal{L})^\infty$ with $z(0) = z$ and $\mathrm{Re}\,z(\sigma) \to \infty$ as $\sigma \to \infty$. By Corollary 4.5, $\mathrm{Sp}(\mathbb{A}_{z(\sigma)}(\pm\infty)) \cap i\mathbb{R} = \emptyset$. By Corollary 4.8, the operators $\mathcal{L}^{\pm} - z(\sigma)I$ are invertible. Therefore each $z(\sigma) \in \rho_{\mathrm{F}}(\mathcal{L}^-) \cap \rho_{\mathrm{F}}(\mathcal{L}^+)$. Therefore $z \in (\rho_{\mathrm{F}}(\mathcal{L}^-) \cap \rho_{\mathrm{F}}(\mathcal{L}^+))^\infty$.

Conversely, let $z \in (\rho_{\mathrm{F}}(\mathcal{L}^-) \cap \rho_{\mathrm{F}}(\mathcal{L}^+))^\infty$. Let $z(\sigma)$ be a path in $(\rho_{\mathrm{F}}(\mathcal{L}^-) \cap \rho_{\mathrm{F}}(\mathcal{L}^+))^\infty$ with $z(0) = z$ and $\mathrm{Re}\,z(\sigma) \to \infty$ as $\sigma \to \infty$. Then the operators $\mathcal{L}^{\pm} - z(\sigma)I$ are invertible, so by Corollary 4.8, for all σ, $\mathrm{Sp}(\mathbb{A}_{z(\sigma)}(\pm\infty)) \cap i\mathbb{R} = \emptyset$. Therefore the number of eigenvalues of $\mathbb{A}_{z(\sigma)}(-\infty)$ (respectively $\mathbb{A}_{z(\sigma)}(\infty)$) with negative real part is independent of σ. Now since $\mathcal{L}$ is the generator of a C_0-semigroup, Proposition 2.4 (1) implies that for $\mathrm{Re}\,z(\sigma)$ large, $z(\sigma) \in \rho_{\mathrm{F}}(\mathcal{L})$. Hence, by Corollary 4.5, for these values of σ, $\mathbb{A}_{z(\sigma)}(-\infty)$ and $\mathbb{A}_{z(\sigma)}(\infty)$ have the same number of eigenvalues with negative real part. Then for all σ, $\mathbb{A}_{z(\sigma)}(-\infty)$ and $\mathbb{A}_{z(\sigma)}(\infty)$ have the same number of eigenvalues with negative real part, so by Corollary 4.5, $z(\sigma) \in \rho_{\mathrm{F}}(\mathcal{L})$ for all σ. Therefore $z \in \rho_{\mathrm{F}}(\mathcal{L})^\infty$. □

We are now ready to prove Proposition 3.2.

For definiteness, assume $\max\{\mathrm{s_F}(\mathcal{L}^-), \mathrm{s_F}(\mathcal{L}^+)\} = \mathrm{s_F}(\mathcal{L}^+)$. We will show that $\mathrm{s_F}(\mathcal{L}) = \mathrm{s_F}(\mathcal{L}^+)$.

If $\mathrm{Re}\,z > \mathrm{s_F}(\mathcal{L}^+) = \max\{\mathrm{s_F}(\mathcal{L}^-), \mathrm{s_F}(\mathcal{L}^+)\}$, then $z \in (\rho_{\mathrm{F}}(\mathcal{L}^-) \cap \rho_{\mathrm{F}}(\mathcal{L}^+))^\infty$. By Proposition 4.9, $z \in \rho_{\mathrm{F}}(\mathcal{L})$.

On the other hand, by the definition of $\mathrm{s_F}(\mathcal{L}^+)$, there exist $z \in \mathrm{Sp_F}(\mathcal{L}^+)$ with $\mathrm{Re}\,z \le \mathrm{s_F}(\mathcal{L}^+)$ but arbitrarily close to $\mathrm{s_F}(\mathcal{L}^+)$. By Corollary 4.8, $\mathrm{Sp}(\mathbb{A}_z(\infty)) \cap i\mathbb{R}$ is nonempty. Then by Corollary 4.5, $z \in \mathrm{Sp_F}(\mathcal{L})$.

By the previous two paragraphs, $\mathrm{s_F}(\mathcal{L}) = \mathrm{s_F}(\mathcal{L}^+)$. This completes the proof.

5. The operators $\mathcal{A}^{\pm}$, $\mathcal{G}^{\pm}$, $\mathcal{L}^{\pm}$, $\mathcal{A}$, and $\mathcal{G}$

Proposition 5.1. *Assume that* $\mathcal{E}_0 = L^2(\mathbb{R})$ *or* $BUC(\mathbb{R})$, *and that Hypothesis 2.1 (a) holds. Let* $\mathcal{C}$ *be any of the constant-coefficient operators* $\mathcal{A}^{\pm}$, $\mathcal{G}^{\pm}$, $\mathcal{L}^{\pm}$. *Then* $\mathrm{s}(\mathcal{C}) = \mathrm{s_{ess}}(\mathcal{C}) = \mathrm{s_F}(\mathcal{C})$, *and these numbers are the same for* $\mathcal{E}_0 = L^2(\mathbb{R})$ *or* $\mathcal{E}_0 = BUC(\mathbb{R})$.

Proof. For $\mathcal{L}^{\pm}$, this is an immediate consequence of Corollary 4.8. The other cases are proved similarly, using the theorem of Sandstede and Scheel or its analog for the case $N_1 = 0$, and Palmer's Theorem. □

Note that Corollary 4.8 can be used to calculate the numbers in Proposition 5.1.

Proposition 5.2. *Assume that Hypothesis 2.1 (a) holds. Let $\mathcal{C}$ be any of the constant-coefficient operators $\mathcal{A}^{\pm}$, $\mathcal{G}^{\pm}$, $\mathcal{L}^{\pm}$, acting respectively on $L^2(\mathbb{R})^{N_1}$, $L^2(\mathbb{R})^{N_2}$, and $L^2(\mathbb{R})^{N_1+N_2}$. Then*

$$\mathrm{s}(\mathcal{C}) = \mathrm{s}_{\mathrm{ess}}(\mathcal{C}) = \mathrm{s}_{\mathrm{F}}(\mathcal{C}) = \omega(\mathcal{C}) = \omega_{\mathrm{ess}}(\mathcal{C}). \tag{5.1}$$

Proof. Fourier transform converts $\mathcal{C}$ into a multiplication operator, for which the result is known, see [7, Propositions I.4.2, I.4.10, IV.3.13]. □

For $\mathcal{C} = \mathcal{L}^{\pm}$, we do not know an easy proof that (5.1) holds on $BUC(\mathbb{R})^{N_1+N_2}$; it is, however, a consequence of our main result, Theorem 3.1. On the other hand, as we shall now see, for $\mathcal{C} = \mathcal{A}^{\pm}$ and $\mathcal{G}^{\pm}$, (5.1) is easily seen to hold on $BUC(\mathbb{R})^{N_1}$ and $BUC(\mathbb{R})^{N_2}$ respectively.

A *sector* is a subset $\Sigma = \Sigma(m, a)$ of $\mathbb{C}$, with $m \in \mathbb{R}$ and $a > 0$, defined by

$$\Sigma(m, a) = \{z = x + iy : |y| < a(m - x)\}.$$

A closed, densely defined linear operator $\mathcal{C}$ is *sectorial* if there is a constant b and a sector $\Sigma(m, a)$ such that $\sigma(\mathcal{C}) \subset \Sigma$, and, for each $z = x + iy \notin \Sigma$, $\|(\mathcal{C} - zI)^{-1}\| \le \frac{b}{|z-m|}$. For the following fact see [7, Corollary IV.3.12].

Proposition 5.3. *For any sectorial operator $\mathcal{C}$, $\mathrm{s}(\mathcal{C}) = \omega(\mathcal{C})$ and $\mathrm{s}_{\mathrm{ess}}(\mathcal{C}) = \omega_{\mathrm{ess}}(\mathcal{C})$.*

The operators $\mathcal{A}^{\pm}$ on $L^2(\mathbb{R})$ or $BUC(\mathbb{R})$ are sectorial and constant-coefficient; therefore, from Propositions 5.2 and 5.3, for $\mathcal{C} = \mathcal{A}^{\pm}$ on $BUC(\mathbb{R})^{N_1}$, (5.1) holds.

Let $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$. An operator on $\mathcal{E}_0^{N_2}$ of the form $\tilde{\mathcal{G}}v = \partial_\xi v + \tilde{B}(\xi)v$, where $\tilde{B}(\xi)$ is a bounded continuous matrix-valued function, generates an *evolutionary semigroup* on $\mathcal{E}_0^{N_2}$ [3]. We have the following fact [3]:

Proposition 5.4. *For any operator $\tilde{\mathcal{G}}$ that generates an evolutionary semigroup, $\mathrm{Sp}(\tilde{\mathcal{G}})$ consists of vertical lines, and (5.1) holds for $\mathcal{C} = \tilde{\mathcal{G}}$.*

Since the operator $c^{-1}\mathcal{G}^{\pm} = \partial_\xi + c^{-1}B_{22}^{\pm}$ generates an evolutionary semigroup on $\mathcal{E}_0^{N_2}$, it follows easily that on $BUC(\mathbb{R})^{N_2}$, (5.1) holds for $\mathcal{C} = \mathcal{G}^{\pm}$.

For the operators $\mathcal{A}$ and $\mathcal{G}$ we record the following facts.

Proposition 5.5. *Assume that $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, and that Hypothesis 2.1 (a) holds. Then*

(1) *The operator $\mathcal{A}$ is* sectorial.
(2) *There is a constant b and a sector Σ such that $\sigma(\mathcal{A}) \subset \Sigma$, and, for each $z = x+iy \notin \Sigma$,*

$$\|(\mathcal{A} - zI)^{-1}\| \le \frac{b}{|y|}. \tag{5.2}$$

(3) $\omega_{\mathrm{ess}}(\mathcal{A}) = \mathrm{s}_{\mathrm{ess}}(\mathcal{A}) = \mathrm{s}_{\mathrm{F}}(\mathcal{A}) = \max\{\mathrm{s}_{\mathrm{F}}(\mathcal{A}^-), \mathrm{s}_{\mathrm{F}}(\mathcal{A}^+)\}$.

The operator $c^{-1}\mathcal{G} = \partial_\xi + c^{-1}B_{22}$ generates an evolutionary semigroup; therefore

Proposition 5.6. *Assume that $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, and that Hypothesis 2.1 (a) holds. Then*

(1) $\mathrm{Sp}(\mathcal{G})$ *consists of vertical lines.*
(2) $\omega(\mathcal{G}) = \omega_{\mathrm{ess}}(\mathcal{G}) = \mathrm{s}(\mathcal{G}) = \mathrm{s}_{\mathrm{ess}}(\mathcal{G}) = \mathrm{s}_{\mathrm{F}}(\mathcal{G}) = \max\{\mathrm{s}_{\mathrm{F}}(\mathcal{G}^-), \mathrm{s}_{\mathrm{F}}(\mathcal{G}^+)\}$.

(3) *If $z \in \rho(\mathcal{G})$, then $z + i\alpha \in \rho(\mathcal{G})$ for all $\alpha \in \mathbb{R}$, and, moreover,*

$$\|(\mathcal{G} - zI)^{-1}\| = \|(\mathcal{G} - (z + i\alpha)I)^{-1}\| = \|(\mathcal{G} - \operatorname{Re} zI)^{-1}\|. \tag{5.3}$$

Proof. (1) and (2) follow from Proposition 5.4. To show (3), we recall that $\mathcal{G} = c\partial_\xi + B_{22}$, with $c \neq 0$. Let us fix any $\alpha \in \mathbb{R}$ and consider the unitary operator $\mathcal{M}_\alpha$ defined for $u \in \mathcal{E}_0^{N_2}$ by the rule $(\mathcal{M}_\alpha u)(\xi) = e^{\frac{\alpha}{c} i\xi} u$. Then

$$\mathcal{G}\mathcal{M}_\alpha = \mathcal{M}_\alpha(\mathcal{G} + i\alpha I), \tag{5.4}$$

yielding (3). □

Remark 5.7. There is no analogue to formula (5.4) if the number c in (2.1) is replaced by a diagonal matrix with different diagonal entries.

Remark 5.8. Formula (5.4) also implies (1). Since it also holds with $\mathcal{G}$ replaced by $\mathcal{G}_\pm$, it implies directly that $\operatorname{Sp}(\mathcal{G}_\pm)$ consists of vertical lines (cf. Proposition 5.4).

Remark 5.9. Statement (2) of Proposition 5.6 says in part that $\max\{s_F(\mathcal{G}^-), s_F(\mathcal{G}^+)\} = s_F(\mathcal{G}) = s(\mathcal{G})$. The first equality follows from Proposition 3.2. The fact that $\mathcal{G}$ has no spectrum to the right of $s_F(\mathcal{G})$ follows from Palmer's Theorem (Theorem 4.2) and the related results of Section 4. Consider the equation

$$\partial_\xi v = -c^{-1}(B_{22}(\xi) - zI)v, \tag{5.5}$$

with $c > 0$ for definiteness. The matrices $-c^{-1}(B_{22}^\pm - zI)$ for $\operatorname{Re} z$ large both have all positive eigenvalues, so all solutions of (5.5) decay exponentially at $\xi = -\infty$ and grow exponentially at $\xi = \infty$. Then this is also true for all z to the right of $s_F(\mathcal{G})$, so no z to the right of $s_F(\mathcal{G})$ can be in $\operatorname{Sp}(\mathcal{G})$.

6. Greiner Spectral Mapping Theorem

The Gearhart-Prüss Theorem, an abstract spectral mapping theorem for C_0-semigroups on Hilbert spaces, says that $e^z \in \rho(e^{\mathcal{L}})$ provided $z + 2\pi ik \in \rho(\mathcal{L})$ for all $k \in \mathbb{Z}$ and $\sup_{k\in\mathbb{Z}} \|(\mathcal{L} - (z + 2\pi ik)I)^{-1}\| < \infty$; see, e.g., [34, Theorem 2.2.4]. However, this theorem does not hold on Banach spaces. Thus, in order give a proof of Proposition 3.3 that includes $\mathcal{E}_0 = BUC(\mathbb{R})$, we have to appeal to the Greiner Spectral Mapping Theorem, which is an abstract spectral mapping theorem for C_0-semigroups on Banach spaces; see, e.g., [34, Theorem 2.2.1], and [18, 19] for more recent discussions.

We recall that a series $\sum_{k\in\mathbb{Z}} w_k$, with $w_k \in \mathcal{X}$, is called Cesaro summable if the following limit exists in $\mathcal{X}$:

$$(C,1)\sum_{k\in\mathbb{Z}} w_k := \lim_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n}\sum_{|k|\le m} w_k = \lim_{n\to\infty}\sum_{k=-n}^{n}\Big(1 - \frac{|k|}{n+1}\Big)w_k. \tag{6.1}$$

We now state Greiner's Theorem.

Theorem 6.1. *Suppose $\mathcal{C}$ is the generator of a C_0-semigroup $\{e^{t\mathcal{C}}\}_{t\ge0}$ on a Banach space $\mathcal{X}$, and let $z \in \mathbb{C}$. Then the following are equivalent:*

(i) $e^z \in \rho(e^{\mathcal{C}})$.
(ii) *$z+2\pi ik \in \rho(\mathcal{C})$ for all $k \in \mathbb{Z}$, and for each $w \in \mathcal{X}$ the series $\sum_{k\in\mathbb{Z}}(\mathcal{C}-(z+2\pi ik)I)^{-1}w$ is Cesaro summable.*

Our proof of Proposition 3.3 will require the following consequence of Greiner's Theorem.

Lemma 6.2. *Let $\mathcal{C}$ be the generator of a C_0-semigroup $\mathcal{T}(t)$ on a Banach space $\mathcal{X}$. Assume (1) 1 is an isolated point of $\mathrm{Sp}(\mathcal{T}(2\pi))$; (2) there is a vertical strip Π around the imaginary axis and an integer $K > 0$ such that $\mathrm{Sp}(\mathcal{C}) \cap \Pi \subset \{in : |n| \le K\}$ and such that $\mathrm{Sp}(\mathcal{C}) \cap \Pi \subset \mathrm{Sp}_{\mathrm{d}}(\mathcal{C})$; and (3) for each $w \in \mathcal{X}$ the series $\sum_{|n|\ge K}(\mathcal{C} - inI)^{-1}w$ is Cesaro summable in $\mathcal{X}$. For the operator $\mathcal{T}(2\pi)$ let $\mathcal{P}$ be the Riesz projection corresponding to the element 1 of its spectrum. Then the range of $\mathcal{P}$ has finite dimension.*

Proof. We employ the idea of the proof of [10, Prop. 1.10 (c)]. Since $\mathrm{Sp}(\mathcal{C}) \cap \Pi \subset \mathrm{Sp}_{\mathrm{d}}(\mathcal{C})$, by Proposition 2.4 (2) we have $1 \in \mathrm{Sp}_p(\mathcal{T}(2\pi))$. The bounded operators $\mathcal{T}(t)$ and $\mathcal{P}$ commute for all $t \ge 0$. Let $\mathcal{X}_1 = \mathrm{rg}\,\mathcal{P}$, $\mathcal{X}_2 = \ker \mathcal{P}$, and denote by $\mathcal{T}_l(t)$ the restriction of $\mathcal{T}(t)$ to $\mathcal{X}_l$, $l = 1, 2$. Let $\mathcal{C}_l = \mathcal{C}|\mathcal{X}_l$, with domain $\mathrm{dom}\,\mathcal{C} \cap \mathcal{X}_l$, be the restriction of $\mathcal{C}$ to $\mathcal{X}_l$, which is the generator of the semigroup $\mathcal{T}_l(t)$ on $\mathcal{X}_l$; see, e.g., [7, Corollary II.2.3]. By the definition of the Riesz projection we have $\mathrm{Sp}(\mathcal{T}_1(2\pi)) = \{1\}$ and $1 \notin \mathrm{Sp}(\mathcal{T}_2(2\pi))$.

Since $\mathrm{Sp}(\mathcal{C}) = \mathrm{Sp}(\mathcal{C}_1) \cup \mathrm{Sp}(\mathcal{C}_2)$, and, by Proposition 2.4 (2), $e^{2\pi\,\mathrm{Sp}(\mathcal{C}_l)} \subset \mathrm{Sp}(\mathcal{T}_l(2\pi))$, we see that

$$\mathrm{Sp}(\mathcal{C}_1) = \mathrm{Sp}_{\mathrm{d}}(\mathcal{C}_1) \subset \{in : |n| < K\}. \tag{6.2}$$

For the operator $\mathcal{C}_1$ and $|n| < K$, let $\mathcal{P}_n$ denote the Riesz projection corresponding to in if $in \in \mathrm{Sp}_{\mathrm{d}}(\mathcal{C}_1)$, and let $\mathcal{P}_n = 0$ if $in \in \rho(\mathcal{C}_1)$. We denote by $\mathcal{P}^0 = \sum_{|n|<K} \mathcal{P}_n$ the Riesz projection for $\mathcal{C}_1$ corresponding to the entire spectrum of $\mathcal{C}_1$. By the properties of Riesz projections, see [13, Section III.6.4], the range of the projection $\mathcal{P}^0$ belongs to the domain of $\mathcal{C}_1$; also, $\mathcal{P}^0$ and $\mathcal{C}_1$ commute, that is, if $x \in \mathrm{dom}\,\mathcal{C}_1$ then $\mathcal{P}^0\mathcal{C}_1 x = \mathcal{C}_1\mathcal{P}^0 x$.

We claim that

$$\ker \mathcal{P}^0 = \{0\}. \tag{6.3}$$

This yields the result, since then $\mathrm{rg}\,\mathcal{P}^0 = \mathcal{X}_1 = \mathrm{rg}\,\mathcal{P}$, and $\mathrm{rg}\,\mathcal{P}^0$ is clearly finite-dimensional.

To prove (6.3) by contradiction, let $\mathcal{X}_0 = \ker \mathcal{P}^0$, a Banach space, and suppose that $\mathcal{X}_0 \ne \{0\}$. Let $\mathcal{C}_0 = \mathcal{C}_1|\mathcal{X}_0$.

Since $\mathcal{P}^0$ is a Riesz projection for the operator $\mathcal{C}_1$, by [13, Section III.6.4] we have

$$\mathrm{Sp}\,\mathcal{C}_0 = \mathrm{Sp}(\mathcal{C}_1|\ker \mathcal{P}^0) = \mathrm{Sp}(\mathcal{C}_1) \setminus \mathrm{Sp}(\mathcal{C}_1|\,\mathrm{rg}\,\mathcal{P}^0) = \emptyset. \tag{6.4}$$

The operator $\mathcal{C}_0$ generates the semigroup $\mathcal{T}_0(t)$, the restriction of the semigroup $\mathcal{T}_1(t)$ to $\mathcal{X}_0$. Then $\mathrm{Sp}(\mathcal{T}_0(2\pi)) \subset \mathrm{Sp}(\mathcal{T}_1(2\pi)) = \{1\}$. Since the operator $\mathcal{T}_0(2\pi)$ is bounded, its spectrum is not empty, and therefore

$$\mathrm{Sp}(\mathcal{T}_0(2\pi)) = \{1\}. \tag{6.5}$$

By condition (3) in the lemma, for each $x \in \mathcal{X}_0$ the series

$$\sum_{|n|\ge K} (\mathcal{C} - inI)^{-1}x = \sum_{|n|\ge K} (\mathcal{C}_0 - inI)^{-1}x$$

is Cesaro summable. By (6.4), the operators $(\mathcal{C}_0 - inI)^{-1}$ exist for $|n| < K$. Therefore for each $x \in \mathcal{X}_0$ the series $\sum_{n\in\mathbb{Z}} (\mathcal{C}_0 - inI)^{-1}x$ is Cesaro summable. Applying Theorem 6.1 for the semigroup $e^{t\mathcal{C}_0} = \mathcal{T}_0(t)$, we conclude that $1 \in \rho(\mathcal{T}_0(2\pi))$. This contradicts (6.5), so (6.3) is proved. □

Remark 6.3. If $\mathcal{X}$ is a Hilbert space, then condition (3) in the lemma can be replaced by $\sup_{|n|\ge K} \|(\mathcal{C} - inI)^{-1}\| < \infty$. In the last paragraph of the proof, this implies that $\sup_{|n|\ge K} \|(\mathcal{C}_0 - inI)^{-1}\| < \infty$. Since the operators $(\mathcal{C}_0 - inI)^{-1}$ exist for $|n| < K$, we have that $\sup_{n\in\mathbb{Z}} \|(\mathcal{C}_0 - inI)^{-1}\| < \infty$. Then the Gearhart-Prüss Theorem implies that $1 \in \rho(\mathcal{T}_0(2\pi))$, which contradicts (6.5) as before.

7. Proof of Proposition 3.3

Before giving the proof of Proposition 3.3 we shall give several preliminary results.

Lemma 7.1. *Assume that $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, and that Hypothesis 2.1 (a) holds. Then $\mathrm{s_F}(\mathcal{G}^\pm) \le \mathrm{s_F}(\mathcal{L}^\pm)$ and $\mathrm{s_F}(\mathcal{G}) \le \mathrm{s_F}(\mathcal{L})$.*

The second conclusion of Lemma 7.1 follows from the first using Proposition 3.2 and Proposition 5.6 (2). The first part of Lemma 7.1 will be proved in Section 8.

Lemma 7.2. *Assume that $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, and that Hypothesis 2.1 (a) holds. Then there is a continuous nonnegative function $r(x)$ defined for $x > \mathrm{s_F}(\mathcal{G})$ such that if $|y| \ge r(x)$ then $z = x + iy \in \rho(\mathcal{L})$. Moreover, for each $x > \mathrm{s_F}(\mathcal{G})$, $\sup_{|y|\ge r(x)} \|(\mathcal{L} - (x+iy)I)^{-1}\| < \infty$.*

Lemma 7.2 will also be proved in Section 8.

Corollary 7.3. *Assume that $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, and that Hypothesis 2.1 (a) holds. Then $\mathrm{s_{ess}}(\mathcal{L}) = \mathrm{s_F}(\mathcal{L})$.*

Proof. Since $\mathrm{s_F}(\mathcal{L}) \le \mathrm{s_{ess}}(\mathcal{L})$ by Proposition 2.4 (3), we only need to show that $\mathrm{s_{ess}}(\mathcal{L}) \le \mathrm{s_F}(\mathcal{L})$. Since $\mathcal{L}$ generates a C_0-semigroup, there is a real number M such that $\{\lambda : \operatorname{Re}\lambda > M\} \subset \rho(\mathcal{L})$. Let $\mathrm{s_F}(\mathcal{L}) < \omega < M$. Then for $\operatorname{Re}\lambda \ge \omega$, $\mathcal{L} - \lambda I$ is Fredholm of index 0. Since $\mathrm{s_F}(\mathcal{G}) \le \mathrm{s_F}(\mathcal{L})$ by Lemma 7.1, Lemma 7.2 implies that $\mathrm{Sp}(\mathcal{L}) \cap \{\lambda : \operatorname{Re}\lambda > \omega\}$ is contained in the compact set $C = \{x + iy : \omega \le x \le M \text{ and } |y| \le r(x)\}$. If $\mathrm{Sp}(\mathcal{L}) \cap C$ were an infinite set, there would exist a point λ_0 in C such that every deleted neighborhood of λ_0 contained both points of $\rho(\mathcal{L})$ and points λ of $\mathrm{Sp}(\mathcal{L})$ at which $\mathcal{L} - \lambda I$ is Fredholm of index 0. This would contradict Theorem IV.5.31 of [13]. Therefore $\mathrm{Sp}(\mathcal{L}) \cap C$ is finite. □

Lemma 7.4. *Assume that $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, and that Hypothesis 2.1 (a) holds. Then for each z with $\operatorname{Re} z > \mathrm{s_F}(\mathcal{G})$, there is an integer $K \ge 0$ such that $z + 2\pi ik \in \rho(\mathcal{L})$ for all $k \in \mathbb{Z}$ with $|k| \ge K$. Moreover,*

$$\text{for each } w \in \mathcal{E}_0, \text{ the series } \sum_{|k|\ge K} (\mathcal{L} - (z + 2\pi ik)I)^{-1} w \text{ is Cesaro summable.} \tag{7.1}$$

Lemma 7.4 will be proved in Section 9.

Corollary 7.5. *Assume that $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, and that Hypothesis 2.1 (a) holds. If $\operatorname{Re} z > \mathrm{s_F}(\mathcal{G})$ and $z + 2\pi ik \in \rho(\mathcal{L})$ for all $k \in \mathbb{Z}$, then $e^z \in \rho(e^{\mathcal{L}})$.*

Proof. The result follows from Lemma 7.4 and the Greiner Spectral Mapping Theorem 6.1. □

For the case $\mathcal{E}_0 = L^2(\mathbb{R})$, Corollary 7.5 follows more easily from Lemma 7.2 and the Gearhart-Prüss Spectral Mapping Theorem.

After these preliminaries, we shall now give the proof of Proposition 3.3.

From Proposition 2.4 (5), $e^{\omega_{\mathrm{ess}}(\mathcal{L})}$ is the radius of the essential spectrum of the operator $e^{\mathcal{L}}$. Thus, we need to show that if $z \in \mathbb{C}$ has $\operatorname{Re} z > \max\{\mathrm{s_F}(\mathcal{L}^+), \mathrm{s_F}(\mathcal{L}^-)\}$ then

$$e^z \in \rho(e^{\mathcal{L}}) \cup \mathrm{Sp_d}(e^{\mathcal{L}}). \tag{7.2}$$

Let $z \in \mathbb{C}$ with $\operatorname{Re} z > \max\{\mathrm{s_F}(\mathcal{L}^+), \mathrm{s_F}(\mathcal{L}^-)\}$. Then $\operatorname{Re} z > \mathrm{s_F}(\mathcal{L})$ by Proposition 3.2, so $\operatorname{Re} z > \mathrm{s_F}(\mathcal{G})$ by Lemma 7.1.

Suppose $z + 2\pi ik \in \rho(\mathcal{L})$ for all $k \in \mathbb{Z}$. Then Corollary 7.5 says that $e^z \in \rho(e^{\mathcal{L}})$, so (7.2) is proved in this case.

Finally, suppose that the intersection of the set $\{z + 2\pi ik : k \in \mathbb{Z}\}$ and $\mathrm{Sp}(\mathcal{L})$ is not empty. From Corollary 7.3, (a) the intersection of the set $\{z + 2\pi ik : k \in \mathbb{Z}\}$ and $\mathrm{Sp}(\mathcal{L})$ consists of points in $\mathrm{Sp}_{\mathrm{d}}(\mathcal{L})$; (b) it contains only finitely many points; and (c) there is a thin vertical strip Π that includes these points but does not contain any other points of $\mathrm{Sp}(\mathcal{L})$. By Proposition 2.4 (2), $e^z \in \mathrm{Sp}_p(e^{\mathcal{L}})$. From Corollary 7.5 we see that e^z is an isolated point of $\mathrm{Sp}(e^{\mathcal{L}})$.

To complete the proof of (7.2), it remains to show that the spectral projection of $e^{\mathcal{L}}$ that corresponds to e^z is finite-dimensional.

This follows from Lemma 6.2 applied to the operator $\mathcal{C} = \frac{1}{2\pi}(\mathcal{L} - zI)$.

8. Proofs of Lemmas 7.1 and 7.2

We introduce the operator families

$$\mathcal{H}(z) = \mathcal{G} - zI - B_{21}(\mathcal{A} - zI)^{-1}B_{12}, \quad z \in \rho(\mathcal{A}); \tag{8.1}$$

$$\mathcal{F}(z) = (\mathcal{G} - zI)^{-1}B_{21}(\mathcal{A} - zI)^{-1}B_{12}, \quad z \in \rho(\mathcal{A}) \cap \rho(\mathcal{G}); \tag{8.2}$$

$$\mathcal{Q}(z) = \mathcal{A} - zI - B_{12}(\mathcal{G} - zI)^{-1}B_{21}, \quad z \in \rho(\mathcal{G}). \tag{8.3}$$

Lemma 8.1. *If $z \in \rho(\mathcal{A})$, then $z \in \rho(\mathcal{L})$ if and only if $\mathcal{H}(z)$ is invertible. In this case the resolvent of $\mathcal{L}$ is the block operator matrix $(\mathcal{L} - zI)^{-1} = (\mathcal{R}_{ij}(z))_{i,j=1}^2$, where:*

$$\mathcal{R}_{11}(z) = (\mathcal{A} - zI)^{-1} + (\mathcal{A} - zI)^{-1}B_{12}\mathcal{H}(z)^{-1}B_{21}(\mathcal{A} - zI)^{-1}, \tag{8.4}$$

$$\mathcal{R}_{12}(z) = -(\mathcal{A} - zI)^{-1}B_{12}\mathcal{H}(z)^{-1}, \tag{8.5}$$

$$\mathcal{R}_{21}(z) = -\mathcal{H}(z)^{-1}B_{21}(\mathcal{A} - zI)^{-1}, \tag{8.6}$$

$$\mathcal{R}_{22}(z) = \mathcal{H}(z)^{-1}. \tag{8.7}$$

Proof. For $z \in \rho(\mathcal{A})$, we have

$$\begin{aligned} \mathcal{L} - zI &= \begin{pmatrix} \mathcal{A} - zI & 0 \\ B_{21} & I \end{pmatrix} \begin{pmatrix} I & (\mathcal{A} - zI)^{-1}B_{12} \\ 0 & \mathcal{H}(z) \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ B_{21}(\mathcal{A} - zI)^{-1} & \mathcal{H}(z) \end{pmatrix} \begin{pmatrix} \mathcal{A} - zI & B_{12} \\ 0 & I \end{pmatrix}. \end{aligned} \tag{8.8}$$

The lemma follows from the triangular structure of the block operator matrices in (8.8). □

Since $\mathcal{A}$ is sectorial, there exist a constant b and a continuous nonnegative function $r(x)$ such that for each $x \in \mathbb{R}$, if $|y| \geq r(x)$ then $z = x + iy \in \rho(\mathcal{A})$ and $\|(\mathcal{A} - zI)^{-1}\| \leq \frac{b}{|y|}$.

Lemma 8.2. *For each $x \in \mathbb{R}$, the following two assertions are equivalent:*

$$x + iy \in \rho(\mathcal{L}) \text{ for } |y| \geq r(x), \text{ and } \sup_{|y| \geq r(x)} \|(\mathcal{L} - (x + iy)I)^{-1}\| < \infty; \tag{8.9}$$

$$\mathcal{H}(x + iy) \text{ is invertible for } |y| \geq r(x), \text{ and } \sup_{|y| \geq r(x)} \|\mathcal{H}(x + iy)^{-1}\| < \infty. \tag{8.10}$$

If the continuous function $r(x)$ is taken sufficiently large, then the equivalent assertions (8.9) and (8.10) hold if and only if $x \in \rho(\mathcal{G})$.

Proof. The equivalence of (8.9) and (8.10) follows from formulas (8.8) and (8.4)–(8.7).

If $x \in \rho(\mathcal{G})$ then, using Proposition 5.6 (3), we conclude that $z = x + iy \in \rho(\mathcal{G})$ for all $y \in \mathbb{R}$ and $\sup_{y \in \mathbb{R}} \|(\mathcal{G} - (x + iy)I)^{-1}\| < \infty$. Now $\mathcal{H}(z) = (\mathcal{G} - zI)(I - \mathcal{F}(z))$ by (8.1). Using

Proposition 5.5 (2) and Proposition 5.6 (3), we conclude that $\|\mathcal{F}(z)\|$ decays as $|y| \to \infty$. Thus, if $r(x)$ is large enough, then $\|\mathcal{F}(z)\| \le 1/2$, yielding

$$\mathcal{H}(z)^{-1} = (I - \mathcal{F}(z))^{-1}(\mathcal{G} - zI)^{-1} = \Big(\sum_{n=0}^{\infty}(\mathcal{F}(z))^n\Big)(\mathcal{G} - zI)^{-1} \tag{8.11}$$

and, eventually, (8.10) using Proposition 5.6 (3) again.

Conversely, if (8.10) holds then (8.1) yields

$$\mathcal{G} - zI = \mathcal{H}(z)(I - \widetilde{\mathcal{F}}(z)) \text{ with } \widetilde{\mathcal{F}}(z) = -\mathcal{H}(z)^{-1}B_{21}(\mathcal{A} - zI)^{-1}B_{12}. \tag{8.12}$$

Now $\|\widetilde{\mathcal{F}}(z)\|$ decays by (8.10) and Proposition 5.5 (2), and thus $\mathcal{G} - zI$ is invertible for large enough $|y|$, yielding $x \in \rho(\mathcal{G})$ by Proposition 5.6 (3). □

Lemma 8.3. *If $z \in \rho(\mathcal{G})$, then $z \in \rho(\mathcal{L})$ if and only if the operator $\mathcal{Q}(z)$ is invertible.*

Proof. For $z \in \rho(\mathcal{G})$ we have

$$\begin{aligned}\mathcal{L} - zI &= \begin{pmatrix} \mathcal{Q}(z) & B_{12}(\mathcal{G} - zI)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ B_{21} & \mathcal{G} - zI \end{pmatrix} \\ &= \begin{pmatrix} I & B_{12} \\ 0 & \mathcal{G} - zI \end{pmatrix} \begin{pmatrix} \mathcal{Q}(z) & 0 \\ (\mathcal{G} - zI)^{-1}B_{21} & I \end{pmatrix}.\end{aligned} \tag{8.13}$$

The lemma follows from the triangular structure of the block operator matrices in (8.13). □

We shall now prove the first conclusion of Lemma 7.1, since, as already noted, the second conclusion follows from the first.

If we replace $\mathcal{G}$, $\mathcal{A}$, and B_{ij} by $\mathcal{G}^-$, $\mathcal{A}^-$, and B_{ij}^-, then in (8.1)–(8.2) we can define operators $\mathcal{H}^-(z)$ and $\mathcal{F}^-(z)$, and in (8.4)–(8.7) we can define operators $\mathcal{R}_{ij}^-(z)$. If, in addition, we replace $\mathcal{L}$ by $\mathcal{L}^-$, and replace the function $r(x)$ by its analog $r^-(x)$ for the operator $\mathcal{A}^-$, the analogs of Lemmas 8.1 and 8.2 remain true.

First, let $\mathcal{E}_0 = L^2(\mathbb{R})$. By Proposition 5.2, $s_F(\mathcal{L}^-) = \omega(\mathcal{L}^-)$. By [7, Theorem I.1.10], if $x > \omega(\mathcal{L}^-)$, then $\sup_{\operatorname{Re} z \ge x} \|(\mathcal{L}^- - zI)^{-1}\| < \infty$. Hence assertion (8.9), with $\mathcal{L}$ replaced by $\mathcal{L}^-$ and $r(x)$ replaced by $r^-(x)$, holds for $x > \omega(\mathcal{L}^-)$. Then the analog of Lemma 8.2 just discussed implies that if $x > \omega(\mathcal{L}^-)$, then $x \in \rho(\mathcal{G}^-)$. Hence by Proposition 5.6, if $\operatorname{Re} z > \omega(\mathcal{L}^-)$, then $z \in \rho(\mathcal{G}^-)$. Therefore $s_F(\mathcal{G}^-) \le s_F(\mathcal{L}^-)$ for $\mathcal{E}_0 = L^2(\mathbb{R})$. But then by Proposition 5.1, $s_F(\mathcal{G}^-) \le s_F(\mathcal{L}^-)$ for $\mathcal{E}_0 = BUC(\mathbb{R})$.

Similarly, $s_F(\mathcal{G}^+) \le s_F(\mathcal{L}^+)$ for $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$.

Finally, we shall prove Lemma 7.2.

Let the function $r(x)$ be defined as above. Let $z = x + iy$ with $x > s_F(\mathcal{G})$ and $|y| \ge r(x)$. Since $x > s_F(\mathcal{G})$, by Proposition 5.6, $z \in \rho(\mathcal{G})$. Since $|y| \ge r(x)$, $z \in \rho(\mathcal{A})$. Therefore we can consider $\mathcal{Q}(z) = (\mathcal{A} - zI)(I - \mathcal{J}(z))$ with $\mathcal{J}(z) = (\mathcal{A} - zI)^{-1}B_{12}(\mathcal{G} - zI)^{-1}B_{21}$. Since $\|\mathcal{J}(z)\|$ decays as $|y| \to \infty$ by Proposition 5.6 (3) and Proposition 5.5 (2), after increasing the continuous function $r(x)$ if necessary, we conclude that $\mathcal{Q}(z)$ is invertible, and thus $z \in \rho(\mathcal{L})$. The estimate on $\|(\mathcal{L} - (x + iy)I)^{-1}\|$ follows from Lemma 8.1.

9. The proof of Lemma 7.4.

This section is devoted to the proof of Lemma 7.4. Fix $z = x + iy$ such that $x > s_F(\mathcal{G})$. For each $k \in \mathbb{Z}$ let $z_k = z + 2\pi ik$. Then for each $k \in \mathbb{Z}$, $z_k \in \rho(\mathcal{G})$. By Lemma 7.2 there is a number K such that if $|k| \ge K$ then $z_k \in \rho(\mathcal{L})$. Since $\mathcal{A}$ is sectorial, we may assume that K

has been chosen so that if $|k| \geq K$ then $z_k \in \rho(\mathcal{A})$. Then for $|k| \geq K$, formulas (8.4)–(8.7) hold. In order to prove Lemma 7.4, we must show that for each $u \in \mathcal{E}_0^{N_1}$, $v \in \mathcal{E}_0^{N_2}$, and $j = 1, 2$, the series $\sum_{|k|\geq K} \mathcal{R}_{j1}(z_k)u$ and $\sum_{|k|\geq K} \mathcal{R}_{j2}(z_k)v$ are Cesaro summable.

First we discuss the series $\sum_{|k|\geq K} \mathcal{R}_{11}(z_k)u$. Since $x \in \rho(\mathcal{G})$, Lemma 8.2 implies that $\sup_k \|\mathcal{H}(z_k)^{-1}\| < \infty$. Then by Proposition 5.5 (2),

$$\|(\mathcal{A} - z_k I)^{-1} B_{12}\mathcal{H}(z_k)^{-1} B_{21}(\mathcal{A} - z_k I)^{-1}\| = O(|k|^{-2}) \text{ as } |k| \to \infty.$$

Hence the series corresponding to the second term in (8.4) converges absolutely. To show that the series $\sum_{|k|\geq K} \mathcal{R}_{11}(z_k)u$ is Cesaro summable, it remains to prove that the series $\sum_{|k|\geq K}(\mathcal{A} - z_k I)^{-1}u$ is Cesaro summable. Fix a large positive ω such that $\omega + x > \mathrm{s}(\mathcal{A})$. Then $\omega + z_k \in \rho(\mathcal{A})$ for all $k \in \mathbb{Z}$ and $e^{\omega+z_k} \in \rho(e^{\mathcal{A}})$ by the spectral mapping theorem for the analytic semigroup generated by $\mathcal{A}$. By the assertion "(i) implies (ii)" in Theorem 6.1 applied to this semigroup, we conclude that the series $\sum_{|k|\geq K}(\mathcal{A} - (\omega + z_k)I)^{-1}u$ is Cesaro summable. By the resolvent identity,

$$(\mathcal{A} - z_k I)^{-1} = (\mathcal{A} - (\omega + z_k)I)^{-1} - \omega(\mathcal{A} - (\omega + z_k)I)^{-1}(\mathcal{A} - z_k I)^{-1}.$$

Since $\mathcal{A}$ is sectorial,

$$\|(\mathcal{A} - (\omega + z_k)I)^{-1}(\mathcal{A} - z_k I)^{-1}\| = O(|k|^{-2}) \text{ as } |k| \to \infty.$$

Therefore the series $\sum_{|k|\geq K}(\mathcal{A} - (\omega + z_k)I)^{-1}(\mathcal{A} - z_k I)^{-1}u$ converges absolutely, so the series $\sum_{|k|\geq K}(\mathcal{A} - z_k I)^{-1}u$ is Cesaro summable as required.

Next we consider the series $\sum_{|k|\geq K} \mathcal{R}_{22}(z_k)v$. Using formulas (8.7) and (8.11), we write

$$\mathcal{R}_{22}(z_k) = \mathcal{H}(z_k)^{-1} = (\mathcal{G} - z_k I)^{-1} + \mathcal{R}_{22}^{(1)}(z_k) + \mathcal{R}_{22}^{(2)}(z_k), \tag{9.1}$$

where

$$\mathcal{R}_{22}^{(1)}(z_k) = \mathcal{F}(z_k)(\mathcal{G} - z_k I)^{-1} \quad \text{and} \quad \mathcal{R}_{22}^{(2)}(z_k) = \sum_{n=2}^{\infty}(\mathcal{F}(z_k))^n(\mathcal{G} - z_k I)^{-1}.$$

Since the spectral mapping theorem holds for evolution semigroups [3] and $z_k \in \rho(\mathcal{G})$ for all k, it follows that $e^z \in \rho(e^{\mathcal{G}})$. Then by the assertion "(i) implies (ii)" in Theorem 6.1 applied to the semigroup generated by $\mathcal{G}$, the series $\sum_{|k|\geq K}(\mathcal{G} - z_k I)^{-1}v$ is Cesaro summable. This takes care of the first summand in (9.1).

Next, we consider the series $\sum_{|k|\geq K} \mathcal{R}_{22}^{(2)}(z_k)v$. Using Proposition 5.5 (2), Proposition 5.6 (3), and (8.2), we have

$$\|\mathcal{F}(z_k)\| = O(|k|^{-1}) \text{ as } |k| \to \infty. \tag{9.2}$$

Recall that $\|\mathcal{F}(z_k)\| \leq 1/2$ for $|k| \geq K$ provided K is large enough; cf. (8.11). Combining these facts, we infer:

$$\sum_{|k|\geq K} \sum_{n=2}^{\infty} \|(\mathcal{F}(z_k))^2(\mathcal{F}(z_k))^{n-2}(\mathcal{G} - z_k I)^{-1}\| \leq c \sum_{|k|\geq K} \|\mathcal{F}(z_k)\|^2 \sum_{n=2}^{\infty} \|\mathcal{F}(z_k)\|^n < \infty.$$

Therefore,

$$\text{the series} \sum_{|k|\geq K} \mathcal{R}_{22}^{(2)}(z_k)v \text{ converges absolutely.} \tag{9.3}$$

This takes care of the last summand in (9.1).

It remains to show that the series

$$\sum_{|k|\geq K}\mathcal{R}_{22}^{(1)}(z_k)v=\sum_{|k|\geq K}(\mathcal{G}-z_kI)^{-1}B_{21}(\mathcal{A}-z_kI)^{-1}B_{12}(\mathcal{G}-z_kI)^{-1}v \tag{9.4}$$

is Cesaro summable. As in the argument for $\sum_{|k|\geq K}\mathcal{R}_{11}(z_k)u$, we can pass in (9.4) from $(\mathcal{A}-z_kI)^{-1}$ to $(\mathcal{A}-(\omega+z_k)I)^{-1}$ with a large $\omega>\mathrm{s}(\mathcal{A})$, so that $\omega+z_k\in\rho(\mathcal{A})$ for all $k\in\mathbb{Z}$. This will also allow us to pass in (9.4) from the series $\sum_{|k|\geq K}$ to the series $\sum_{k\in\mathbb{Z}}$ because $z_k\in\rho(\mathcal{G})$ for all $k\in\mathbb{Z}$. Obviously, $e^z=e^{z_k}$. As before, $e^{\omega+z_k}\in\rho(e^{\mathcal{A}})$ since the analytic semigroup generated by $\mathcal{A}$ has the spectral mapping property, and $e^z\in\rho(e^{\mathcal{G}})$.

We will use the following general formulas for the resolvent of the semigroup generators; cf., e.g., [3, p. 25] or [26, (2.2.7), (2.2.8)]:

$$(\mathcal{G}-z_kI)^{-1}v=\int_0^1 e^{z_k(1-s)}e^{s\mathcal{G}}(e^{\mathcal{G}}-e^zI)^{-1}v\,ds, \tag{9.5}$$

$$(\mathcal{G}-z_kI)^{-1}v=(e^{\mathcal{G}}-e^zI)^{-1}\int_0^1 e^{z_k(1-s_1)}e^{s_1\mathcal{G}}v\,ds_1, \tag{9.6}$$

$$(\mathcal{A}-(\omega+z_k)I)^{-1}v=(e^{\mathcal{A}}-e^{\omega+z}I)^{-1}\int_0^1 e^{z_k(1-s_2)}e^{\omega(1-s_2)}e^{s_2\mathcal{A}}v\,ds_2. \tag{9.7}$$

Taking into account (9.5), we let $w=(e^{\mathcal{G}}-e^zI)^{-1}v$. Taking into account formula (9.6), we note that each term in (9.4) includes left multiplication by the same bounded operator $(e^{\mathcal{G}}-e^zI)^{-1}$, and the presence of this operator does not affect the summability. We conclude that series (9.4) is Cesaro summable if and only if the series $\sum_{k\in\mathbb{Z}}a_k$ is Cesaro summable, where

$$a_k=\int_0^1\int_0^1\int_0^1 e^{z_k(3-s_1-s_2-s)}F(s_1,s_2)f(s)\,ds\,ds_1\,ds_2, \tag{9.8}$$

$$F(s_1,s_2)=e^{s_1\mathcal{G}}B_{21}(e^{\mathcal{A}}-e^{\omega+z}I)^{-1}e^{\omega(1-s_2)}e^{s_2\mathcal{A}}B_{12},\quad s_1,s_2\in[0,1], \tag{9.9}$$

$$f(s)=e^{s\mathcal{G}}w,\quad s\in[0,1]. \tag{9.10}$$

In (9.8)–(9.10) we make the change of variables $\tilde{s}_1=2\pi(1-s_1)$, $\tilde{s}_2=2\pi(1-s_2)$, $\tilde{s}=2\pi s$. We write

$$e^{z_k(3-s_1-s_2-s)}=e^{-(2\pi-\tilde{s}_1)z/2\pi}\cdot e^{-(2\pi-\tilde{s}_2)z/2\pi}\cdot e^{3z}\cdot e^{-\tilde{s}z/2\pi}\cdot e^{ik(\tilde{s}_1+\tilde{s}_2-\tilde{s})}, \tag{9.11}$$

and we introduce functions $\tilde{F}(\tilde{s}_1,\tilde{s}_2)$ and $\tilde{f}(\tilde{s})$ defined by

$$\begin{aligned}\tilde{F}(\tilde{s}_1,\tilde{s}_2)&=e^{-(2\pi-\tilde{s}_1)z/2\pi}F(1-\tilde{s}_1/2\pi,1-\tilde{s}_2/2\pi)e^{-(2\pi-\tilde{s}_2)z/2\pi}\\&=e^{(2\pi-\tilde{s}_1)(G-zI)/2\pi}B_{21}(e^{A}-e^{\omega+z}I)^{-1}e^{\omega\tilde{s}_2/2\pi}e^{(2\pi-\tilde{s}_2)(A-zI)/2\pi}B_{12},\end{aligned} \tag{9.12}$$

$$\tilde{f}(\tilde{s})=f(\tilde{s}/2\pi)e^{3z}e^{-\tilde{s}z/2\pi}=e^{3z}e^{\tilde{s}(G-zI)/2\pi}w,\ \tilde{s}_1,\tilde{s}_2,\tilde{s}\in[0,2\pi]. \tag{9.13}$$

We infer that

$$a_k=\frac{1}{(2\pi)^3}\int_{\mathbb{T}}\int_{\mathbb{T}}\int_{\mathbb{T}}\tilde{F}(\tilde{s}_1,\tilde{s}_2)\tilde{f}(\tilde{s})e^{ik(\tilde{s}_1+\tilde{s}_2-\tilde{s})}\,d\tilde{s}\,d\tilde{s}_1\,d\tilde{s}_2, \tag{9.14}$$

where $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}\simeq[0,2\pi]$ is the torus with the measure $dt/2\pi$. In what follows we drop the tildes in formulas (9.12), (9.13), and (9.14).

We recall from [14, Sec. I.2.5] the definition of Fejer's summability kernel

$$K_n(t) = \sum_{k=-n}^{n} \Big(1 - \frac{|k|}{n+1}\Big) e^{ikt}, \quad t \in \mathbb{T}, \tag{9.15}$$

whose convolution property is that

$$\|K_n * f - f\|_{L^1(\mathbb{T};\mathcal{E}_0^{N_2})} \to 0 \text{ as } n \to \infty \text{ for every } f \in L^1(\mathbb{T};\mathcal{E}_0^{N_2}). \tag{9.16}$$

Using (6.1) and (9.14), we represent the sum $(C,1)\sum_{k\in\mathbb{Z}} a_k$ in terms of convolution with K_n:

$$(C,1)\sum_{k\in\mathbb{Z}} a_k = \lim_{n\to\infty} \frac{1}{(2\pi)^2} \int\limits_{\mathbb{T}^2} F(s_1,s_2) \frac{1}{2\pi} \int\limits_{\mathbb{T}} K_n(s_1+s_2-s) f(s)\, ds\, ds_1\, ds_2. \tag{9.17}$$

Clearly $f \in L^1(\mathbb{T};\mathcal{E}_0^{N_2})$. We let $\|F\|_\infty = \sup_{(s_1,s_2)\in\mathbb{T}} \|F(s_1,s_2)\|$. We can write the last integral in (9.17) as the convolution $(K_n * f)(s_1+s_2)$. We now claim that

$$(C,1)\sum_{k\in\mathbb{Z}} a_k = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} F(s_1,s_2) f(s_1+s_2)\, ds_1\, ds_2, \tag{9.18}$$

which yields Cesaro summability of the series $\sum_{|k|\geq K} R_{22}^{(1)}(z_k)v$ in (9.4). Indeed, (9.18) follows from (9.17), (9.16), and the estimate

$$\begin{aligned}
\Big\| \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} & F(s_1,s_2) \Big[(K_n * f)(s_1+s_2) - f(s_1+s_2)\Big]\, ds_1\, ds_2 \Big\|_{\mathcal{E}_0^{N_2}} \\
&\leq \frac{\|F\|_\infty}{(2\pi)^2} \int_{\mathbb{T}} \Big(\int_{\mathbb{T}} \|(K_n * f)(s_1+s_2) - f(s_1+s_2)\|_{\mathcal{E}_0^{N_2}}\, ds_1 \Big)\, ds_2 \\
&= \frac{\|F\|_\infty}{(2\pi)^2} \int_{\mathbb{T}} \Big(\int_{\mathbb{T}} \|(K_n * f)(s_1) - f(s_1)\|_{\mathcal{E}_0^{N_2}}\, ds_1 \Big)\, ds_2 \\
&= \|F\|_\infty \|K_n * f - f\|_{L^1(\mathbb{T};\mathcal{E}_0^{N_2})}.
\end{aligned}$$

Finally, Cesaro summability of the series $\sum_{|k|\geq K} \mathcal{R}_{12}(z_k)v$ and $\sum_{|k|\geq K} \mathcal{R}_{21}(z_k)u$ are proved similarly, and, in fact, more easily, as there are only two integrals in the formulas analogous to (9.8)–(9.10). Indeed, let us outline the steps in the proof of Cesaro summability of the series $\sum_{|k|\geq K} \mathcal{R}_{12}(z_k)v$. By (8.5) and (9.1), we need to show that the following three series are Cesaro summable:

$$\sum_{|k|\geq K} (\mathcal{A} - z_k I)^{-1} B_{12} (\mathcal{G} - z_k I)^{-1} v, \tag{9.19}$$

$$\sum_{|k|\geq K} (\mathcal{A} - z_k I)^{-1} B_{12} \mathcal{F}(z_k) (\mathcal{G} - z_k I)^{-1} v, \tag{9.20}$$

$$\sum_{|k|\geq K} (\mathcal{A} - z_k I)^{-1} B_{12} \mathcal{R}_{22}^{(2)}(z_k) v. \tag{9.21}$$

Using (9.3) and $\|(\mathcal{A}-z_k I)^{-1}\| = O(|k|^{-1})$, we conclude that series (9.21) converges absolutely. Using Proposition 5.6 (3) and (9.2), we conclude that the norms of the summands of series (9.20) are $O(|k|^{-2})$, proving its summability. It remains to consider (9.19). Using formulas

(9.5) – (9.7), we conclude that (9.19) is Cesaro summable if and only if the series $\sum_{k\in\mathbb{Z}} a_k$ is Cesaro summable, where

$$a_k = \int_0^1 e^{z_k(2-s_1-s)} F(s_1) f(s)\, ds\, ds_1, \tag{9.22}$$

$$F(s_1) = e^{\omega(1-s_1)} e^{s_1\mathcal{A}} B_{12}(e^{\mathcal{G}} - e^{z})^{-1}, \quad s_1 \in [0,1], \tag{9.23}$$

$$f(s) = e^{s\mathcal{G}} v, \quad s \in [0,1]. \tag{9.24}$$

The remainder of the argument, using Fejer's summability kernel, is similar to the argument given above.

This completes the proof of Lemma 7.4.

10. Stability of FitzHugh–Nagumo fronts

We consider the FitzHugh–Nagumo equation

$$u_t = u_{xx} + f(u) - v, \tag{10.1}$$

$$v_t = \epsilon(u - \gamma v), \tag{10.2}$$

with $x \in \mathbb{R}$, $f(u) = u(1-u)(u-a)$, $a \in (0,1)$ fixed. This equation is a simplification of the Hodgkin-Huxley equation, which models of propagation of electrical waves along nerve axons.

To study traveling waves with speed c, we replace x by $\xi = x - ct$ and obtain

$$u_t = u_{\xi\xi} - cu_\xi + f(u) - v, \tag{10.3}$$

$$v_t = -cv_\xi + \epsilon(u - \gamma v), \tag{10.4}$$

Traveling waves are stationary solutions of (10.3)–(10.4), i.e., $u_t = v_t = 0$. The existence of various traveling fronts $(u^*(\xi), v^*(\xi))$, for which $\lim_{\xi\to\pm\infty}(u^*(\xi), v^*(\xi))$ both exist but are different, has been shown by Yanagida [35] and Deng [6]. The fronts approach their limits exponentially as $\xi \to \pm\infty$. Also, they have $c \neq 0$, so that Hypothesis (H) is satisfied.

Let $(u^*(\xi), v^*(\xi))$ be a stationary front solution of (10.3)–(10.4), so that $(u^*(x-ct), v^*(x-ct))$ is a traveling front solution of (10.1)–(10.2). Writing $(u,v) = (u^*, v^*) + (U, V)$ and then $W = (U, V)$, (10.1)–(10.2) becomes

$$W_t = \mathcal{L}W + \mathcal{N}(W) \tag{10.5}$$

with

$$\mathcal{L}W = \mathcal{L}(U,V) = \begin{pmatrix} \partial_{\xi\xi} - c\partial_\xi + f'(u^*(\xi)) & -1 \\ \epsilon & -c\partial_\xi - \epsilon\gamma \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix},$$

$$\mathcal{N}(W) = \mathcal{N}(U,V) = \begin{pmatrix} f(u^*(\xi) + U(\xi)) - f(u^*(\xi)) - f'(u^*(\xi))U(\xi) \\ 0 \end{pmatrix}.$$

The linear operator $\mathcal{L}$ is bounded on $\mathcal{E}_0^2$ for $\mathcal{E}_0 = BUC(\mathbb{R})$, $L^2(\mathbb{R})$, or $H^1(\mathbb{R})$. The nonlinear operator $\mathcal{N}$ is C^1 on $\mathcal{E}_0^2$ for $\mathcal{E}_0 = BUC(\mathbb{R})$ or $H^1(\mathbb{R})$.

The traveling wave $(u^*(\xi), v^*(\xi))$ is *asymptotically stable with asymptotic phase* in $\mathcal{E}_0^2$ if for each $\epsilon > 0$ there is a number $\delta > 0$ such that for each $(U_0, V_0) \in \mathcal{E}_0^2$ with $\|(U_0, V_0)\| < \delta$, the solution $(u(\xi,t), v(\xi,t))$ of (10.3)–(10.4) with $(u(\xi,0), v(\xi,0)) = (u^*(\xi), v^*(\xi)) + (U_0(\xi), V_0(\xi))$ has the following properties:

(1) There is a function $\sigma : [0,\infty) \to \mathbb{R}$ such that $\|(u(\xi,t), v(\xi,t) - (u^*(\xi+\sigma(t)), v^*(\xi + \sigma(t))\| < \epsilon$ for all $t \geq 0$.

(2) There is a number σ with $|\sigma| < \epsilon$ such that $\|(u(\xi,t),v(\xi,t))-(u^*(\xi+\sigma),v^*(\xi+\sigma))\| \to 0$ as $t \to \infty$.

Theorem 10.1. *Let $\mathcal{E}_0 = BUC(\mathbb{R})$ or $H^1(\mathbb{R})$. Then each of the traveling front solutions $(u^*(\xi),v^*(\xi))$, $\xi = x - ct$, for (10.1)–(10.2) whose existence was shown in Yanagida [35] or Deng [6] is asymptotically stable with asymptotic phase in $\mathcal{E}_0^2$.*

Proof. For $\mathcal{E}_0 = BUC(\mathbb{R})$ or $L^2(\mathbb{R})$, Yanagida [35], Nii [23], and Sandstede [30] show that $s_F(\mathcal{L}) < 0$, the only element of $\mathrm{Sp}(\mathcal{L})$ in $\mathrm{Re}\,\lambda \geq 0$ is 0, and 0 is a simple eigenvalue. With the aid of [9], Section 4.2, it follows that the same is true for $\mathcal{E}_0 = H^1(\mathbb{R})$. By Corollary 3.5, $e^{t\mathcal{L}}$ is linearly stable on $\mathcal{E}_0^2$ for $\mathcal{E}_0 = BUC(\mathbb{R})$, $L^2(\mathbb{R})$, or $H^1(\mathbb{R})$. For $\mathcal{E}_0 = BUC(\mathbb{R})$ or $H^1(\mathbb{R})$, the mapping $\mathcal{N}$ is C^1, and $D\mathcal{N}(0) = 0$. Then by the theorem of Bates and Jones [1], for $\mathcal{E}_0 = BUC(\mathbb{R})$ or $H^1(\mathbb{R})$, $(u^*(\xi),v^*(\xi))$ is asymptotically stable with asymptotic phase in $\mathcal{E}_0^2$. □

11. Discussion

As was mentioned in Section 2, Hypothesis (H), which states that $c \neq 0$, is not necessary. The reason is that, while the operator $c\partial_\xi$, $c \neq 0$, has spectrum equal to the imaginary axis, and hence is not sectorial, the operator $0\partial_\xi = 0$ has spectrum equal to $\{0\}$, and *is* sectorial. Thus if $c = 0$, so the traveling wave is a standing wave, then the diagonal operator (2.5) is sectorial; therefore the operator $\mathcal{L}$ is also sectorial, since it is a bounded perturbation of a sectorial operator. In this case Theorem 3.1 is a consequence of Proposition 5.5.

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Appendix A. Proof of Theorem 4.1

We have $(\mathcal{L} - zI)(u, v) = h = (h_1, h_2)$ if and only if

$$\begin{aligned} du'' + au' + (B_{11} - zI)u + B_{12}v = h_1, \\ cv' + B_{21}u + (B_{22} - zI)v - zv = h_2, \end{aligned} \tag{A.1}$$

and $\mathcal{T}_z(u,w,v) = g = (g_1,g_2,g_3)$ if and only if

$$\begin{aligned} u' - w &= g_1, \\ w' + d^{-1}aw + d^{-1}(B_{11} - zI)u + d^{-1}B_{12}v &= g_2, \\ v' + c^{-1}B_{21}u + c^{-1}(B_{22} - zI)v &= g_3. \end{aligned} \tag{A.2}$$

It follows that $(u,v) \in \ker(\mathcal{L} - zI)$ if and only if $(u,u',v) \in \ker \mathcal{T}_z$ (so $\dim\ker(\mathcal{L} - zI) = \dim\ker \mathcal{T}_z I$), and also that

$$(h_1,h_2) \in \operatorname{rg}(\mathcal{L} - zI) \text{ if and only if } (0, d^{-1}h_1, c^{-1}h_2) \in \operatorname{rg}\mathcal{T}_z. \tag{A.3}$$

If $\operatorname{rg}\mathcal{T}_z$ is closed, then by (A.3), $\operatorname{rg}(\mathcal{L} - zI)$ is closed. To show the converse, we define the invertible operator $\mathcal{T}_{\mathrm{ref}} = \partial_\xi - \left[\begin{smallmatrix} 0 & I & 0 \\ d^{-1} & 0 & 0 \\ 0 & 0 & c^{-1} \end{smallmatrix}\right]$ on $\mathcal{E}_0^{N_1} \times \mathcal{E}_0^{N_1} \times \mathcal{E}_0^{N_2}$, and the bounded operator

$$\mathcal{P} = \begin{pmatrix} I + B_{11} - zI & a & B_{12} \\ -B_{21} & 0 & I + B_{22} - zI \end{pmatrix} \mathcal{T}_{\mathrm{ref}}^{-1} \tag{A.4}$$

from $\mathcal{E}_0^{N_1} \times \mathcal{E}_0^{N_1} \times \mathcal{E}_0^{N_2}$ to $\mathcal{E}_0^{N_1} \times \mathcal{E}_0^{N_2}$. We claim that

$$g \in \operatorname{rg}\mathcal{T}_z \text{ if and only if } \mathcal{P}g \in \operatorname{rg}(\mathcal{L} - zI). \tag{A.5}$$

Assuming the claim, suppose $\operatorname{rg}(\mathcal{L} - zI)$ is closed. Suppose $g^{(n)} \in \operatorname{rg}\mathcal{T}_z$ and $g^{(n)} \to g$ as $n \to \infty$. Then $\mathcal{P}g^{(n)} \in \operatorname{rg}(\mathcal{L} - zI)$ by (A.5), and $\mathcal{P}g^{(n)} \to \mathcal{P}g$ because $\mathcal{P}$ is continuous. Since $\operatorname{rg}(\mathcal{L} - zI)$ is closed, $\mathcal{P}g \in \operatorname{rg}(\mathcal{L} - zI)$. But then $g \in \operatorname{rg}\mathcal{T}_z$ by (A.5). Therefore $\operatorname{rg}\mathcal{T}_z$ is closed. Thus $\operatorname{rg}\mathcal{T}_z$ is closed if and only if $\operatorname{rg}(\mathcal{L} - zI)$ is closed.

To prove (A.5), let $\tilde{\mathcal{P}} = \left[\begin{smallmatrix} d^{-1} & 0 \\ 0 & c^{-1} \end{smallmatrix}\right]\mathcal{P}$. By (A.3), $\mathcal{P}g \in \operatorname{rg}(\mathcal{L} - zI)$ if and only if $(0, \tilde{\mathcal{P}}g) \in \operatorname{rg}\mathcal{T}_z$. Let $U, g \in \mathcal{E}_0^{2N_1+N_2}$, and let $V = U - \mathcal{T}_{\mathrm{ref}}^{-1}g$. Then $\mathcal{T}_z U = g$ if and only if

$$\mathcal{T}_z V = (\mathcal{T}_{\mathrm{ref}} - \mathcal{T}_z)\mathcal{T}_{\mathrm{ref}}^{-1} g = (0, \tilde{\mathcal{P}}g).$$

Claim (A.5) follows.

In the remainder of the proof we assume that $\operatorname{rg}\mathcal{T}_z$ and $\operatorname{rg}(\mathcal{L} - zI)$ are closed. Let $[g]_{\mathcal{T}}$ (respectively, $[h]_{\mathcal{L}}$) denote an element of the quotient space $\mathcal{E}_0^{2N_1+N_2}/\operatorname{rg}\mathcal{T}_z$ (respectively, $\mathcal{E}_0^{N_1+N_2}/\operatorname{rg}(\mathcal{L} - zI)$). We claim that

$$n := \dim(\mathcal{E}_0^{N_1+N_2}/\operatorname{rg}(\mathcal{L} - zI)) \le \dim(\mathcal{E}_0^{2N_1+N_2}/\operatorname{rg}\mathcal{T}_z). \tag{A.6}$$

To see this, choose $h^{(j)} \in \mathcal{E}_0^{N_1+N_2}$, $j = 1, \dots, n$, such that $[h^{(j)}]_{\mathcal{L}}$ are linearly independent, and let $g^{(j)} = (0, d^{-1}h_1^{(j)}, c^{-1}h_2^{(j)})$. If

$$\sum_{j=1}^n \alpha_j [g^{(j)}]_{\mathcal{T}} = [\sum_{j=1}^n \alpha_j g^{(j)}]_{\mathcal{T}} = [0]_{\mathcal{T}},$$

then $\sum_{j=1}^n \alpha_j g^{(j)} \in \operatorname{rg}\mathcal{T}_z$, so $\sum_{j=1}^n \alpha_j h^{(j)} \in \operatorname{rg}(\mathcal{L} - zI)$ by (A.3), and hence all $\alpha_j = 0$. This yields (A.6).

Finally, to show that the inequality (A.6) is in fact an equality, pick any $[g^{(j)}]_{\mathcal{T}}$, $j = 1, \dots, n+1$. Since $n = \dim(\mathcal{E}_0^{N_1+N_2}/\operatorname{rg}(\mathcal{L} - zI))$, the $[\mathcal{P}g^{(j)}]_{\mathcal{L}}$ are linearly dependent. Therefore there are constants α_j not all zero such that $\sum_{j=1}^{n+1} \alpha_j [\mathcal{P}g^{(j)})]_{\mathcal{L}} = [0]_{\mathcal{L}}$. It follows that

$\mathcal{P}(\sum_{j=1}^{n+1} \alpha_j g^{(j)}) \in \operatorname{rg}(\mathcal{L}-zI)$, so by (A.5), $\sum_{j=1}^{n+1} \alpha_j g^{(j)} \in \operatorname{rg}\mathcal{T}_z$. Thus $[g^{(j)}]_{\mathcal{T}}$, $j = 1,\dots,n+1$, are linearly dependent, and so $n = \dim(\mathcal{E}_0^{2N_1+N_2}/\operatorname{rg}\mathcal{T}_z)$, as required.

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