Heteroclinic solutions of a singularly perturbed Hamiltonian system

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Motivation for work


The physical motivation comes from a multi-order-parameter phase field model, developed by Braun et al. for the description of crystalline interphase boundaries. The smallness of $\varepsilon$ is related to large anisotropy. [The heteroclinic orbit represents a moving interface between ordered and disordered states.] The mathematical interest stems from the fact that the smoothness and normal hyperbolicity of the critical manifold fails at certain points. Thus the well-developed geometric singular perturbation theory does not apply. The existence of such a heteroclinic, and its dependence on $\varepsilon$, is proved via a functional analytic approach.

Motivation for talk

Show how the blow-up technique of geometric singular perturbation theory (Du-mortier, Roussarie, Szmolyan, Krupa, . . . ) can help with such problems.

Help is: geometric matching of outer and inner solutions.
Second-order system

We consider

\[(1)\]  \[x_{tt} = g_x(x,y),\]
\[(2)\]  \[\varepsilon^2 y_{tt} = g_y(x,y),\]

where

\[(3)\]  \[g(x,y) = \frac{1}{4}y^4 - \frac{1}{2}xy^2 + h(x).\]
First-order system

Write (1)–(2) as a first-order system (the slow system) with \( u_1 = x, \ u_3 = y \):

\[
\begin{align*}
(4) \quad u_1\tau &= u_2, \\
(5) \quad u_2\tau &= g_x(u_1, u_3) = -\frac{1}{2}u_3^2 + h'(u_1), \\
(6) \quad \varepsilon u_3\tau &= u_4, \\
(7) \quad \varepsilon u_4\tau &= g_y(u_1, u_3) = u_3^3 - u_1u_3.
\end{align*}
\]

In (4)–(7) let \( \tau = \varepsilon\sigma \). We obtain the fast system:

\[
\begin{align*}
(8) \quad u_1\sigma &= \varepsilon u_2, \\
(9) \quad u_2\sigma &= \varepsilon g_x(u_1, u_3) = \varepsilon \left(-\frac{1}{2}u_3^2 + h'(u_1)\right), \\
(10) \quad u_3\sigma &= u_4, \\
(11) \quad u_4\sigma &= g_y(u_1, u_3) = u_3^3 - u_1u_3 = u_3(u_3^2 - u_1).
\end{align*}
\]

Equilibria of the fast system for \( \varepsilon > 0 \):

\[
(u_1, 0, 0, 0) \text{ with } h'(u_1) = 0, \quad (u_1, 0, \pm u_1^\frac{1}{2}, 0) \text{ with } -\frac{1}{2}u_1 + h'(u_1) = 0.
\]
Assumptions on $h$:
Equilibria of the fast system

\[ u_{1\sigma} = \varepsilon u_2, \]
\[ u_{2\sigma} = \varepsilon g_x(u_1, u_3) = \varepsilon \left( -\frac{1}{2} u_3^2 + h'(u_1) \right), \]
\[ u_{3\sigma} = u_4, \]
\[ u_{4\sigma} = g_y(u_1, u_3) = u_3^3 - u_1 u_3 = u_3(u_3^2 - u_1) \]

for \( \varepsilon > 0 \):

\[(x_-, 0, 0, 0), \quad (x_0, 0, \pm x_0^{\frac{1}{3}}, 0), \quad (x_+, 0, \pm x_+^{\frac{1}{3}}, 0).\]
For each $\varepsilon$, the fast system has the first integral

$$H(u_1, u_2, u_3, u_4) = \frac{1}{2}u_2^2 + \frac{1}{2}u_4^2 - g(u_1, u_3).$$

Note:

$$H(x_-, 0, 0, 0) = H(x_+, 0, \frac{1}{2}x_+, 0) = 0.$$ 

Goal: show that for small $\varepsilon > 0$, there is a heteroclinic solution of the fast system from $(x_-, 0, 0, 0)$ to $(x_+, 0, \frac{1}{2}x_+, 0)$.

For $\varepsilon > 0$, $(x_-, 0, 0, 0)$ and $(x_+, 0, \frac{1}{2}x_+, 0)$ are hyperbolic equilibria of the fast system with two negative eigenvalues and two positive eigenvalues.

The heteroclinic solution will correspond to an intersection of the 2-dimensional manifolds $W^u_\varepsilon(x_-, 0, 0, 0)$ and $W^s_\varepsilon(x_+, 0, \frac{1}{2}x_+, 0)$ that is transverse within the 3-dimensional manifold $H^{-1}(0)$ (which is indeed a manifold away from equilibria).
Fast limit and slow systems

Set $\varepsilon = 0$ in the fast system to obtain the fast limit system:

\begin{align*}
(12) & \quad u_{1\sigma} = 0, \\
(13) & \quad u_{2\sigma} = 0, \\
(14) & \quad u_{3\sigma} = u_4, \\
(15) & \quad u_{4\sigma} = g_y(u_1, u_3) = u_3(u_3^2 - u_1).
\end{align*}

Equilibria (slow manifold):
Three manifolds of normally hyperbolic equilibria:

\[ E_- = \{ (u_1, u_2, 0, 0) : u_1 < 0 \text{ and } u_2 \text{ arbitrary} \} , \]
\[ F_- = \{ (u_1, u_2, -u_1^{\frac{1}{3}}, 0) : u_1 > 0 \text{ and } u_2 \text{ arbitrary} \} , \]
\[ F_+ = \{ (u_1, u_2, u_1^{\frac{1}{2}}, 0) : u_1 > 0 \text{ and } u_2 \text{ arbitrary} \} . \]

Each has one positive eigenvalue and one negative eigenvalue. (On \( E_+ \) there are two pure imaginary eigenvalues. On the \( u_2 \)-axis all eigenvalues are 0.)
Set $\varepsilon = 0$ in the slow system to obtain the slow limit system:

\begin{align*}
(16) \quad u_{1\tau} &= u_2, \\
(17) \quad u_{2\tau} &= g_x(u_1,u_3) = -\frac{1}{2}u_3^2 + h'(u_1), \\
(18) \quad 0 &= u_4, \\
(19) \quad 0 &= g_y(u_1,u_3) = u_3(u_3^2 - u_1).
\end{align*}

$E_{\pm}, F_{\pm}$ are manifolds of solutions of (18)–(19). Equations (16)–(17) give the slow system on these manifolds.

Slow system on $E_-$ ($u_1 < 0$, $u_2$ arbitrary):

\begin{align*}
(20) \quad u_{1\tau} &= u_2, \\
(21) \quad u_{2\tau} &= g_x(u_1,0) = h'(u_1).
\end{align*}

Slow system on $F_+$ ($u_1 > 0$, $u_2$ arbitrary):

\begin{align*}
(22) \quad u_{1\tau} &= u_2, \\
(23) \quad u_{2\tau} &= g_x(u_1,u_1^{\frac{1}{2}}) = -\frac{1}{2}u_1 + h'(u_1).
\end{align*}
Phase portraits of slow system on $E_-$ and $F_+$ in $u_1u_2$-coordinates, both extended to $u_1 = 0$:

- In (a), $(x_-, 0)$ is a hyperbolic saddle, and a branch of its unstable manifold meets the $u_2$ axis at a point $(0, u_2^*)$.

- In (b), $(x_+, 0)$ is a hyperbolic saddle, and a branch of its stable manifold meets the $u_2$ axis at the same point $(0, u_2^*)$. 
Slow limit system on $E_-$ and $F_+$:

\[ x - u_1, u_2, u_3 - u \]

\[ E_- \quad u_2^* \quad E_+ \]

\[ F_- \quad \Gamma_- \quad \Gamma_+ \quad F_+ \]

\[ x_- \]

\[ u_1 \]

\[ u_3 \]

**Theorem 1.** For small $\varepsilon > 0$, there is a heteroclinic solution of the fast system from $(x_-, 0, 0, 0)$ to $(x_+, 0, x_+^{\frac{1}{2}}, 0)$ that is close to $\Gamma_- \cup \Gamma_+$. 
Blow-up

To the fast system append the equation $\epsilon_\sigma = 0$:

(24) \quad u_{1\sigma} = \epsilon u_2,

(25) \quad u_{2\sigma} = \epsilon g_x(u_1, u_3) = \epsilon \left(-\frac{1}{2}u_3^2 + h'(u_1)\right),

(26) \quad u_{3\sigma} = u_4,

(27) \quad u_{4\sigma} = g_y(u_1, u_3) = u_3^3 - u_1 u_3,

(28) \quad \epsilon_\sigma = 0.

The $u_2$-axis consists of equilibria of (24)–(27) with $\epsilon = 0$ that are not normally hyperbolic within $u_{1u_2u_3u_4}$-space.

In $u_{1u_2u_3u_4}\epsilon$-space, we shall it blow up to the product of the $u_2$-axis with a 3-sphere. The 3-sphere is a blow-up of the origin in $u_{1u_3u_4}\epsilon$-space.

The blowup transformation is a map from $\mathbb{R} \times S^3 \times [0, \infty)$ to $u_{1u_2u_3u_4}\epsilon$-space. Let $(u_2, (\tilde{u}_1, \tilde{u}_3, \tilde{u}_4, \tilde{\epsilon}), \tilde{r})$ be a point of $\mathbb{R} \times S^3 \times [0, \infty)$; we have $\tilde{u}_1^2 + \tilde{u}_3^2 + \tilde{u}_4^2 + \tilde{\epsilon}^2 = 1$. Then

(29) \quad u_1 = \tilde{r}^2 \tilde{u}_1, \quad u_2 = u_2, \quad u_3 = \tilde{r} \tilde{u}_3, \quad u_4 = \tilde{r}^2 \tilde{u}_4, \quad \epsilon = \tilde{r}^3 \tilde{\epsilon}.$
Under this transformation (24)–(28) pulls back to a vector field $X$ on $\mathbb{R} \times S^3 \times [0, \infty)$ for which the cylinder $\bar{r} = 0$ consists entirely of equilibria. The vector field we shall study is $\tilde{X} = \bar{r}^{-1}X$. Division by $\bar{r}$ desingularizes the vector field on the cylinder $\bar{r} = 0$ but leaves it invariant.

Let $p_-(\varepsilon)$ (respectively $p_+(\varepsilon)$) be the unique point in $\mathbb{R} \times S^3 \times [0, \infty)$ that corresponds to $(x_-, 0, 0, 0, \varepsilon)$ (respectively $(x_+, 0, x_+^{\frac{1}{2}}, 0, \varepsilon)$). We wish to show that for small $\varepsilon > 0$ there is an integral curve of $X$ from $p_-(\varepsilon)$ to $p_+(\varepsilon)$. Equivalently, we shall show that for small $\varepsilon > 0$ there is an integral curve of $\tilde{X}$ from $p_-(\varepsilon)$ to $p_+(\varepsilon)$. 
In blow-up space:

- $\tilde{\Gamma}_-$ corresponds to $\Gamma_-$ and approaches a point $\tilde{q}_- = (u^*_2, \hat{q}_-, 0)$ on the blow-up cylinder.
- $\tilde{\Gamma}_+$ corresponds to $\Gamma_+$ and approaches a point $\tilde{q}_+ = (u^*_2, \hat{q}_+, 0)$ on the blow-up cylinder.
- On the blow-up cylinder, each 3-sphere $u_2 = \text{constant}$ is invariant.

**Proposition 2.** There is an integral curve $\tilde{\Gamma}_0$ of $\tilde{X}$ from $\tilde{q}_-$ to $\tilde{q}_+$ that lies in the 3-dimensional hemisphere given by $u_2 = u^*_2$, $\bar{r} = 0$, $\bar{\varepsilon} > 0$.

**Theorem 3.** For small $\varepsilon > 0$ there is an integral curve $\tilde{\Gamma}(\varepsilon)$ of $\tilde{X}$ from $p_-(\varepsilon)$ to $p_+(\varepsilon)$. As $\varepsilon \to 0$, $\tilde{\Gamma}(\varepsilon) \to \tilde{\Gamma}_- \cup \tilde{\Gamma}_0 \cup \tilde{\Gamma}_+$. 
We shall need three charts on blow-up space:
Chart for $\bar{\epsilon} > 0$

On the set of points in $\mathbb{R} \times S^3 \times [0, \infty)$ with $\bar{\epsilon} > 0$, let

$$(30) \quad u_1 = r^2 b_1, \quad u_2 = u_2, \quad u_3 = rb_3, \quad u_4 = r^2 b_4, \quad \epsilon = r^3,$$

with $r \geq 0$. After division by $r$, (24)–(28) becomes

$$(31) \quad b_{1s} = u_2,$$

$$(32) \quad u_{2s} = r^2 \left( -\frac{1}{2} r^2 b_3^2 + h'(r^2 b_1) \right),$$

$$(33) \quad b_{3s} = b_4,$$

$$(34) \quad b_{4s} = b_3^3 - b_1 b_3,$$

$$(35) \quad r_s = 0.$$

**Note 1:** $r = 0$ implies $u_{2s} = 0$.

**Note 2:** $b_1 = \tilde{u}_1 \bar{\epsilon}^{-\frac{2}{3}}, \, u_2, \, b_3 = \tilde{u}_3 \bar{\epsilon}^{-\frac{1}{3}}, \, b_4 = \tilde{u}_4 \bar{\epsilon}^{-\frac{2}{3}},$ and $r = \tilde{r} \bar{\epsilon}^{\frac{1}{3}}$.

**Note 3:** (31)–(35) actually represents the vector field

$$r^{-1}X = \tilde{r}^{-1} \bar{\epsilon}^{-\frac{1}{3}}X = \bar{\epsilon}^{-\frac{1}{3}} \tilde{X}$$
Chart for \( \bar{u}_1 < 0 \)

On the set of points in \( \mathbb{R} \times S^3 \times [0, \infty) \) with \( \bar{u}_1 < 0 \), let

\[ u_1 = -v^2, \quad u_2 = u_2, \quad u_3 = va_3, \quad u_4 = v^2a_4, \quad \varepsilon = v^3 \delta, \]

with \( v \geq 0 \). After division by \( v \), (24)–(28) becomes

\begin{align*}
(37) \quad v_t &= -\frac{1}{2}v\delta u_2, \\
(38) \quad u_{2t} &= v^2\delta(-\frac{1}{2}v^2a_3^2 + h'(-v^2)), \\
(39) \quad a_{3t} &= a_4 + \frac{1}{2}\delta u_2a_3, \\
(40) \quad a_{4t} &= a_3^3 + a_3 + \delta u_2a_4, \\
(41) \quad \delta_t &= \frac{3}{2}\delta^2 u_2.
\end{align*}

**Note 1:** \( v = 0 \) implies \( u_{2t} = 0 \).

**Note 2:** \( v = \bar{r}(\bar{u}_1)^{\frac{1}{2}}, u_2, a_3 = \bar{u}_3(\bar{u}_1)^{-\frac{1}{2}}, a_4 = -\bar{u}_4\bar{u}_1^{-1}, \) and \( \delta = \bar{\varepsilon}(\bar{u}_1)^{-\frac{3}{2}} \).

**Note 3:** (37)–(41) actually represents the vector field

\[ v^{-1}X = \bar{r}^{-1}(\bar{u}_1)^{-\frac{1}{2}}X = (-\bar{u}_1)^{-\frac{1}{2}}\bar{X} \]
Chart for $\bar{u}_1 > 0$

On the set of points in $\mathbb{R} \times S^3 \times [0, \infty)$ with $\bar{u}_1 > 0$, let

\begin{align*}
(42) \quad u_1 &= w^2, \quad u_2 = u_2, \quad u_3 = wc, \quad u_4 = w^2 c_4, \quad \varepsilon = w^3 \gamma.
\end{align*}

with $w \geq 0$. After division by $w$, (24)–(28) becomes

\begin{align*}
(43) \quad w_t &= \frac{1}{2} w \gamma u_2, \\
(44) \quad u_{2t} &= w^2 \gamma (\frac{1}{2} w^2 c_3^2 + h'(w^2)), \\
(45) \quad c_{3t} &= c_4 - \frac{1}{2} \gamma u_2 c_3, \\
(46) \quad c_{4t} &= c_3^3 - c_3 - \gamma u_2 c_4, \\
(47) \quad \gamma_t &= -\frac{3}{2} \gamma^2 u_2.
\end{align*}

Note 1: $w = 0$ implies $u_{2t} = 0$.

Note 2: $w = \bar{r} \bar{u}_1^{\frac{1}{2}}, u_2, c_3 = \bar{u}_3 \bar{u}_1^{-\frac{1}{2}}, c_4 = \bar{u}_4 \bar{u}_1^{-1},$ and $\gamma = \bar{\varepsilon} \bar{u}_1^{-\frac{3}{2}}$.

Note 3: (43)–(47) actually represents the vector field

$$w^{-1}X = \bar{r}^{-1} \bar{u}_1^{-\frac{1}{2}} X = \bar{u}_1^{-\frac{1}{2}} \bar{X}$$
Construction of the inner solution $\tilde{\Gamma}_0$

Let $\hat{X}$ denote the restriction of the vector field $\tilde{X}$ to the invariant 3-sphere $M = \{u_2^*\} \times S^3 \times \{0\}$, $S^3 = \{(\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{e}) : \bar{u}_1^2 + \bar{u}_3^2 + \bar{u}_4^2 + \bar{e}^2 = 1\}$.

Chart on the open subset of $M$ with $\bar{u}_1 < 0$: $a_3 = \bar{u}_3(-\bar{u}_1)^{-\frac{1}{2}}$, $a_4 = -\bar{u}_4\bar{u}_1^{-1}$, $\delta = \bar{e}(-\bar{u}_1)^{-\frac{3}{2}}$. In this chart, the vector field $(-\bar{u}_1)^{-\frac{1}{2}}\hat{X}$ is

\begin{align}
(48) \quad a_{3t} &= a_4 + \frac{1}{2}\delta u_2^*a_3, \\
(49) \quad a_{4t} &= a_3^3 + a_3 + \delta u_2^*a_4, \\
(50) \quad \delta_t &= \frac{3}{2}\delta^2 u_2^*.
\end{align}
Chart on the open subset of $M$ with $\bar{u}_1 > 0$: $c_3 = \bar{u}_3 \bar{u}_1^{-\frac{1}{2}}$, $c_4 = \bar{u}_4 \bar{u}_1^{-1}$, $\gamma = \bar{e} \bar{u}_1^{-\frac{3}{2}}$. In this chart, the vector field $\bar{u}_1^{-\frac{1}{2}} \hat{X}$ is

\begin{align*}
(51) \quad c_{3t} &= c_4 - \frac{1}{2} \gamma u_2^* c_3, \\
(52) \quad c_{4t} &= c_3^3 - c_3 - \gamma u_2^* c_4, \\
(53) \quad \gamma_t &= -\frac{3}{2} \gamma^2 u_2^*.
\end{align*}
Chart on the open subset of $M$ with $\bar{\varepsilon} > 0$: \( b_1 = \bar{u}_1\bar{\varepsilon}^{-\frac{2}{3}}, \) \( b_3 = \bar{u}_3\bar{\varepsilon}^{-\frac{1}{3}}, \) \( b_4 = \bar{u}_4\bar{\varepsilon}^{-\frac{2}{3}}. \) In this chart, the vector field $\bar{\varepsilon}^{-\frac{1}{3}}\hat{X}$ is

\begin{align*}
(54) & \quad b_{1s} = u_2^*, \\
(55) & \quad b_{3s} = b_4, \\
(56) & \quad b_{4s} = b_3^3 - b_1b_3 = b_3(b_3^2 - b_1).
\end{align*}

The solution of (54) with $b_1(0) = 0$ is $b_1 = u_2^*s$. Substitute into (56) and combining (55) and (56) into a second-order equation:

\begin{equation}
(57) \quad b_{3ss} = b_3(b_3^2 - u_2^*s)
\end{equation}

By Sourdis and Fife, (57) has a solution $b_3(s)$ with $b_{3s} > 0$ such that

\begin{align*}
(S1) & \quad b_3(s) = O\left(|s|^{-\frac{1}{4}}e^{-\frac{2}{3}(u_2^*)^\frac{1}{2}|s|\frac{3}{2}}\right) \text{ as } s \to -\infty, \\
(S2) & \quad b_3(s) = (u_2^*)\frac{1}{2} + O(s^{-\frac{5}{2}}) \text{ as } s \to \infty, \\
(S3) & \quad b_{3s}(s) \leq C|s|^{-\frac{1}{2}}, \text{ } s \neq 0.
\end{align*}

$$(u_2^*s, b_3(s), b_{3s}(s))$$ is a solution of (54)–(56). It represents an intersection of $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ in the 3-sphere $M$. 

Transversality

$W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ are 2-dimensional submanifolds of the 3-sphere $M$.

Let $\tilde{\Gamma}_0 = (u_2^*, \hat{\Gamma}_0, 0)$. They intersect along $\hat{\Gamma}_0$.

**Proposition 4.** $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ intersect transversally within $M$ along $\hat{\Gamma}_0$.

**Proof.** The linearization of

\[
\begin{align*}
  b_{1s} &= u_2^*, \\
  b_{3s} &= b_4, \\
  b_{4s} &= b_3^3 - b_1 b_3
\end{align*}
\]

along $(u_2^s, b_3(s), b_{3s}(s))$ is

\[
(58) \quad \begin{pmatrix}
  B_{1s} \\
  B_{3s} \\
  B_{4s}
\end{pmatrix} = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 1 \\
  -b_3(s) & 3b_3(s)^2 - u_2^s & 0
\end{pmatrix} \begin{pmatrix}
  B_1 \\
  B_3 \\
  B_4
\end{pmatrix}.
\]

We must show there are no solutions with appropriate behavior at $s = \pm \infty$ other than multiples of $(u_2^*, b_{3s}, b_{3ss})$. 
There is a complementary 2-dimensional space of solutions of (58) with $B_1(s) = 0$ and $(B_3(s), B_4(s))$ a solution of

\[
\begin{pmatrix}
B_{3s} \\
B_{4s}
\end{pmatrix} =
\begin{pmatrix}
0 & 1 \\
3b_3(s)^2 - u_2^* s & 0
\end{pmatrix}
\begin{pmatrix}
B_3 \\
B_4
\end{pmatrix}
\tag{59}
\]

We must show that no nontrivial solution has appropriate behavior at $s = \pm \infty$.

(59) is equivalent to the second order linear system

\[
B_{3ss} = (3b_3(s)^2 - u_2^* s)B_3.
\tag{60}
\]

Proof of Theorem 3

**Theorem 3.** For small $\varepsilon > 0$ there is an integral curve $\tilde{\Gamma}(\varepsilon)$ of $\tilde{X}$ from $p_-(\varepsilon)$ to $p_+(\varepsilon)$. As $\varepsilon \to 0$, $\tilde{\Gamma}(\varepsilon) \to \tilde{\Gamma}_- \cup \tilde{\Gamma}_0 \cup \tilde{\Gamma}_+$.

Recall: for each $\varepsilon$, the fast system has the first integral

$$H(u_1, u_2, u_3, u_4) = \frac{1}{2} u_2^2 + \frac{1}{2} u_4^2 - \left( \frac{1}{4} u_3^4 - \frac{1}{2} u_1 u_3^2 + h(u_1) \right).$$

$H$ gives rise to a first integral for $\tilde{H}$ on blow-up space:

$$\tilde{H}(u_2, (\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\varepsilon}), \bar{r}) = \frac{1}{2} u_2^2 + \bar{r}^4 \left( \frac{1}{2} \bar{u}_4^2 - \frac{1}{4} \bar{u}_3^4 + \frac{1}{2} \bar{u}_1 \bar{u}_3^2 \right) - h(\bar{r}^2 \bar{u}_1).$$
Let $N_{\varepsilon}$ denote the set of points in blow-up space at which $\tilde{H} = 0$ and $\bar{r}^3 \bar{\varepsilon} = \varepsilon$.

Away from equilibria of $\tilde{X}$, each $N_{\varepsilon}$ is a manifold of dimension 3.

For the vector field $\tilde{X}$ and $\varepsilon > 0$, the equilibria $p_-(\varepsilon)$ and $p_+(\varepsilon)$ have 2-dimensional unstable and stable manifolds.

We will prove the theorem by showing that for small $\varepsilon > 0$, $W^u(p_-(\varepsilon))$ and $W^s(p_+(\varepsilon))$ have a nonempty intersection that is transverse within $N_{\varepsilon}$.
Chart for $\bar{u}_1 < 0$:

\[
\begin{align*}
\nu_t &= -\frac{1}{2}v\delta u_2, \\
\dot{u}_{2t} &= v^2\delta(-\frac{1}{2}v^2a_3^2 + h'(-v^2)), \\
a_{3t} &= a_4 + \frac{1}{2}\delta u_2 a_3, \\
a_{4t} &= a_3^3 + a_3 + \delta u_2 a_4, \\
\delta_t &= \frac{3}{2}\delta^2 u_2.
\end{align*}
\]

The 3-dimensional space $a_3 = a_4 = 0$ is invariant, and is normally hyperbolic near the plane of equilibria $a_3 = a_4 = \delta = 0$. One eigenvalue is positive, one is negative.

The plane of equilibria corresponds to $E_-$. Normal hyperbolicity within $\delta = 0$ is not lost at $v = 0$, which corresponds to $u_1 = 0$.

Restrict to $a_3 = a_4 = 0$ and divide by $\delta$:

\[
\begin{align*}
(61) & \quad \dot{v} = -\frac{1}{2}vu_2, \\
(62) & \quad \dot{u}_2 = v^2h'(-v^2), \\
(63) & \quad \dot{\delta} = \frac{3}{2}\delta u_2.
\end{align*}
\]
\begin{align*}
\dot{v} &= -\frac{1}{2}vu_2, \\
\dot{u}_2 &= v^2 h'(-v^2), \\
\dot{\delta} &= \frac{3}{2} \delta u_2.
\end{align*}

Equilibria on the lines \( \{(v, u_2, \delta) : v = (-x_-)^{1/2}, u_2 = 0\} \) and \( \{(v, u_2, \delta) : v = \delta = 0, u_2 \neq 0\} \) are normally hyperbolic, with one positive eigenvalue and one negative eigenvalue.
**Lemma 4.** As $\delta_0 \to 0+$, $W^u((-x_-)^{1/2}, 0, \delta_0)$ approaches $W^u(0, u_2^*, 0)$ in the $C^1$ topology. (Both have dimension 1.)

**Lemma 5.** In the chart for $\bar{u}_1 < 0$, as $\delta_0 \to 0+$, $W^u((-x_-)^{1/2}, 0, 0, 0, \delta_0)$ approaches the manifold of unstable fibers over $W^u(0, u_2^*, 0)$ in the $C^1$ topology. (Both have dimension 2.)

The latter corresponds to $W^{cu}(\hat{q}_1)$ in $M = \{u_2^*\} \times S^3 \times \{0\}$. 
Chart for $\bar{u}_1 > 0$:

$$
\begin{align*}
  w_t &= \frac{1}{2} w \gamma u_2, \\
  u_{2t} &= w^2 \gamma (-\frac{1}{2} w^2 c_3^2 + h'(w^2)), \\
  c_{3t} &= c_4 - \frac{1}{2} \gamma u_2 c_3, \\
  c_{4t} &= c_3^3 - c_3 - \gamma u_2 c_4, \\
  \gamma_t &= -\frac{3}{2} \gamma^2 u_2.
\end{align*}
$$

The equilibria of the plane $c_3 = 1, c_4 = \gamma = 0$ have, transverse to the plane, one positive eigenvalue, one negative eigenvalue, one zero eigenvalue.

Therefore this plane is part of a 3-dimensional normally hyperbolic invariant manifold $S_2$, with equations

$$
c_3 = 1 + \gamma^2 \tilde{c}_3(w, u_2, \gamma), \quad c_4 = \gamma \tilde{c}_4(w, u_2, \gamma).
$$

The plane of equilibria corresponds to $F_+$. Normal hyperbolicity within $\gamma = 0$ is not lost at $w = 0$, which corresponds to $u_1 = 0$.

Restrict to $S_2$ and divide by $\gamma$: 
(64) \[ w_t = \frac{1}{2} w u_2, \]

(65) \[ u_{2t} = w^2 \left( -\frac{1}{2} w^2 (1 + \gamma^2 \tilde{c}_3)^2 + h'(w^2) \right), \]

(66) \[ \gamma_t = -\frac{3}{2} \gamma u_2. \]

**Lemma 6.** As \( \gamma_0 \to 0^+ \), \( W^s(x_+^{\frac{1}{2}}, 0, \gamma_0) \) approaches \( W^s(0, u_2^*, 0) \) in the \( C^1 \) topology. (Both have dimension 1.)

**Lemma 7.** In the chart for \( \bar{u}_1 > 0 \), as \( \gamma_0 \to 0^+ \), \( W^s(x_+^{\frac{1}{2}}, 0, 1, 0, \gamma_0) \) approaches the manifold of stable fibers over \( W^s(0, u_2^*, 0) \) in the \( C^1 \) topology. (Both have dim 2.)

The latter corresponds to \( W^{cs}(\hat{q}_+) \) in \( M = \{u_2^*\} \times S^3 \times \{0\} \).
In blow-up space:

**Lemma 8.** As $\varepsilon \to 0^+$, $W^u(p_-(\varepsilon))$ approaches $W^{cu}(\hat{q}_-)$ in the $C^1$ topology.

**Lemma 9.** As $\varepsilon \to 0^+$, $W^s(p_+(\varepsilon))$ approaches $W^{cs}(\hat{q}_+)$ in the $C^1$ topology.

By Proposition 4: $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ meet transversally within the 3-sphere $\bar{r} = 0$, $u_2 = u_2^*$, which is $N_0$.

In the chart for $\bar{\varepsilon} > 0$, $H$ corresponds to

$$H_b(b_1, u_2, b_3, b_4, r) = \frac{1}{2}u_2^2 + r^4\left(\frac{1}{2}b_4^2 - \frac{1}{4}b_3^4 + \frac{1}{2}b_1b_3^2\right) + h(r^2b_1).$$

$N_0$ corresponds to the set of $(b_1, u_2, b_3, b_4, r)$ such that $H_b = 0$ and $r = 0$. The functions $H_b$ and $r$ have linearly independent gradients provided $u_2 \neq 0$. Therefore, where $u_2 \neq 0$, the sets $N_{\varepsilon^3} = N_r$ depend smoothly on $r$. Since $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ meet transversally within $N_0$, it follows that $W^u(p_-(\varepsilon))$ and $W^s(p_+(\varepsilon))$ meet transversally within $N_\varepsilon$ for $\varepsilon$ small.