Exchange Lemmas

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Plan

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Boundary Value Problems

\[
\dot{\xi} = F(\xi, \varepsilon), \quad \xi(t_-) \in A_-(\varepsilon), \quad \xi(t_+) \in A_+(\varepsilon),
\]

To show existence of a solution: show that the manifold of solutions that start on \( A_-(\varepsilon) \) and the manifold of solutions that end on \( A_+(\varepsilon) \) meet transversally.

Remarks

- The problem with \( \varepsilon = 0 \) may be degenerate in some major way.
- Such problems are called *singularly perturbed*.
- The geometric approach to these problems, which focuses on *tracking manifolds of potential solutions* rather than on asymptotic expansions of solutions, is called *geometric singular perturbation theory* (Fenichel, Kopell, Jones, ...).
Exchange Lemma of Jones and Kopell

Slow-Fast Systems

\[
\dot{a} = f(a, b, \varepsilon), \quad \dot{b} = \varepsilon g(a, b, \varepsilon), \quad (a, b) \in \mathbb{R}^n \times \mathbb{R}^m.
\]

Set \( \varepsilon = 0 \):

\[
\dot{a} = f(a, b, 0), \quad \dot{b} = 0.
\]

Assume:

1. \( f(\hat{a}(b), b, 0) = 0 \).
2. \( D_a f(\hat{a}(b), b, 0) \) has
   - \( k \) eigenvalues with negative real part.
   - \( l \) eigenvalues with positive real part.
   - \( k + l = n \).
3. \( g(\hat{a}(b), b, 0) \neq 0 \).
After a change of coordinates:

\[
\begin{align*}
\dot{x} &= A(x, y, c, \varepsilon)x, \\
\dot{y} &= B(x, y, c, \varepsilon)y, \\
\dot{c} &= \varepsilon((1, 0, \ldots, 0) + L(x, y, c, \varepsilon)xy),
\end{align*}
\]

\[(x, y, c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m,\]

\(A(0, 0, c, 0)\) has eigenvalues with negative real part,
\(B(0, 0, c, 0)\) has eigenvalues with positive real part.

Flow with \(\varepsilon = 0\).

Flow with \(\varepsilon > 0\).
Exchange Lemma of Jones and Kopell with \( m = 1 \)

Assume:

1. \( m = 1 \) (for simplicity).
2. For each \( \epsilon \), \( M_\epsilon \) is a submanifold of \( xyc \)-space of dimension \( l \).
3. \( M = \{ (x, y, c, \epsilon) : (x, y, c) \in M_\epsilon \} \) is itself a manifold.
4. \( M_0 \) meets \( xc \)-space transversally at a point \( (x_*, 0, 0) \).

Under the forward flow, each \( M_\epsilon \) becomes a manifold \( M_\epsilon^* \) of dimension \( l + 1 \).
**Theorem 1 (Exchange Lemma of Jones and Kopell with \( m = 1, 1994 \)).** Consider a point \((0, y^*, c^*)\) with \(y^* \neq 0\) and \(0 < c^*\). Let \(A\) be a small neighborhood of \((y^*, c^*)\) in \(yc\)-space. Then for small \(\varepsilon_0 > 0\) there is a smooth function \(\tilde{x} : A \times [0, \varepsilon_0) \to \mathbb{R}^k\) such that:

1. \(\tilde{x}(y, c, 0) = 0\).
2. As \(\varepsilon \to 0\), \(\tilde{x} \to 0\) exponentially, along with its derivatives with respect to all variables.
3. For \(0 < \varepsilon < \varepsilon_0\), \(\{(x, y, c) : (y, c) \in A \text{ and } x = \tilde{x}(y, c, \varepsilon)\}\) is contained in \(M_{\varepsilon}^*\).

Transversality to \(xc\)-space is “exchanged” for closeness to \(yc\)-space.
Brunovsky’s Reformulation of Jones and Kopell’s Exchange Lemma as an Inclination Lemma

**Theorem 2 (1999).** Let $0 < c^*$. Let $A$ be a small neighborhood of $(0, c^*)$ in $yc$-space. Then for small $\varepsilon_0 > 0$ there is a smooth function $\tilde{x} : A \times [0, \varepsilon_0) \to \mathbb{R}^k$ such that:

1. $\tilde{x}(y, c, 0) = 0$.
2. As $\varepsilon \to 0$, $\tilde{x} \to 0$ exponentially, along with its derivatives with respect to all variables.
3. For $0 < \varepsilon < \varepsilon_0$, $\{(x, y, c) : (y, c) \in A$ and $x = \tilde{x}(y, c, \varepsilon)\}$ is contained in $M^*_\varepsilon$.

![Diagram (a)](image1.png)

![Diagram (b)](image2.png)

![Diagram (c)](image3.png)
General Exchange Lemma

Important Features of the Exchange Lemma

(1) There is a normally hyperbolic invariant manifold \((c\text{-}space)\) and a small parameter \(\varepsilon\).

(2) There is a collection of submanifolds \(M_\varepsilon\) of \(xyc\)-space such that \(M = \{(x, y, c, \varepsilon) : (x, y, c) \in M_\varepsilon\}\) is itself a manifold. \(M_0\) meets \(xc\)-space transversally in a manifold \(N_0\) (here a point).

(3) \(N_0\) projects along the stable fibration of \(xc\) space to a submanifold \(P_0\) of \(c\)-space of the same dimension (here a point).

(4) For small \(\varepsilon > 0\), the flow on \(c\)-space is followed for a long time.

(5) It takes \(P_\varepsilon\) to a set \(P_\varepsilon^*\) of dimension one greater. As \(\varepsilon \to 0\), the limit of \(P_\varepsilon^* \neq \) where the limiting DE takes \(P_0\). Nevertheless, the limit of \(P_\varepsilon^*\) exists and has the same dimension. Call it \(P_0^*\).

(6) As \(\varepsilon \to 0\), \(M_\varepsilon^* \to W^u(P_0^*)\).
General Exchange Lemma (S., 2007). (1)–(5) plus technical assumptions imply (6).

What’s the point?

- To understand the flow on the normally hyperbolic invariant manifold may require rectification, blowing-up, etc.
- Once you’ve done this work, the General Exchange Lemma helps deal with the remaining dimensions.
Exchange Lemma of Jones and Tin

Consider again:

\[ \dot{x} = A(x, y, c, \varepsilon)x, \]
\[ \dot{y} = B(x, y, c, \varepsilon)y, \]
\[ \dot{c} = \varepsilon((1, 0, \ldots, 0) + L(x, y, c, \varepsilon)xy), \]

\((x, y, c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m,\)

\(A(0, 0, c, 0)\) has eigenvalues with negative real part,
\(B(0, 0, c, 0)\) has eigenvalues with positive real part.

Assume:

(1) \(m \geq 1.\)
(2) For each \(\varepsilon,\) \(M_\varepsilon\) is a submanifold of \(xyc\)-space of dimension \(l + p, 0 \leq p \leq m - 1.\)
(3) \(M = \{(x, y, c, \varepsilon) : (x, y, c) \in M_\varepsilon\}\) is itself a manifold.
(4) \(M_0\) meets \(xc\)-space transversally in a manifold \(N_0\) of dimension \(p.\)
(5) \(N_0\) projects smoothly to a submanifold \(P_0\) of \(c\)-space of dimension \(p.\)
(6) The vector \((1, 0, \ldots, 0)\) is not tangent to \(P_0.\)
Then:

1. Each $M_\varepsilon$ meets $xc$-space transversally in a manifold $N_\varepsilon$ of dimension $p$.
2. $N_\varepsilon$ projects smoothly to a submanifold $P_\varepsilon$ of $c$-space of dimension $p$.
3. The vector $(1, 0, \ldots, 0)$ is not tangent to $P_\varepsilon$.

After a change of coordinates $c = (u, v, w) \in \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^{m-1-p}$ that takes each $P_\varepsilon$ to $v$-space, the system can be put in the form

$$
\begin{align*}
\dot{x} &= A(x, y, u, v, w, \varepsilon)x, \\
\dot{y} &= B(x, y, u, v, w, \varepsilon)y, \\
\dot{u} &= \varepsilon(1 + e(x, y, u, v, w, \varepsilon)xy), \\
\dot{v} &= \varepsilon F(x, y, u, v, w, \varepsilon)xy, \\
\dot{w} &= \varepsilon G(x, y, u, v, w, \varepsilon)xy.
\end{align*}
$$
Under the forward flow, each $M_\varepsilon$ becomes a manifold $M_\varepsilon^*$ of dimension $l + p + 1$. Each $P_\varepsilon$ becomes a manifold $P_\varepsilon^*$ of dimension $p + 1$, which in our coordinates is just $uv$-space.

**Theorem 3 (Exchange Lemma of Jones and Tin).** Let $0 < u^*$. Let $A$ be a small neighborhood of $(0, u^*, 0)$ in $yuv$-space. Then for small $\varepsilon_0 > 0$ there are smooth function $\tilde{x}: A \times [0, \varepsilon_0) \to \mathbb{R}^k$ and $\tilde{w}: A \times [0, \varepsilon_0) \to \mathbb{R}^{m-p-1}$ such that:

1. $\tilde{x}(y, u, v, 0) = 0$.
2. $\tilde{w}(y, u, v, 0) = \tilde{w}(0, u, v, \varepsilon) = 0$.
3. As $\varepsilon \to 0$, $(\tilde{x}, \tilde{w}) \to 0$ exponentially, along with its derivatives with respect to all variables.
4. For $0 < \varepsilon < \varepsilon_0$, $\{(x, y, u, v, w) : (y, u, v) \in A \text{ and } (x, w) = (\tilde{x}, \tilde{w})(y, u, v, \varepsilon)\}$ is contained in $M_\varepsilon^*$.

**Remark**

The theorem also applies to

$$\dot{x} = A(x, y, c, \varepsilon)x,$$
$$\dot{y} = B(x, y, c, \varepsilon)y,$$
$$\dot{c} = \varepsilon(1, 0, \ldots, 0) + L(x, y, c, \varepsilon)xy,$$

It is really about perturbations of systems with a family of normally hyperbolic equilibria, not about slow-fast systems.
Loss-of-Stability Turning Points: Liu’s Exchange Lemma

Liu considers a slow-fast system

\[
\begin{align*}
\dot{a} &= f(a, b, \varepsilon), \\
\dot{b} &= \varepsilon g(a, b, \varepsilon),
\end{align*}
\]

with \(a \in \mathbb{R}^{k+l+1}\) and \(b \in \mathbb{R}^{m-1}, m \geq 2\). Assume:

1. \(f(0, b, \varepsilon) = 0\). (Hence for each \(\varepsilon\), \(b\)-space is invariant, and for \(\varepsilon = 0\) it consists of equilibria.)

2. \(D_a f(0, b, 0)\) has
   - \(k\) eigenvalues with real part less than \(\lambda_0 < 0\);
   - \(l\) eigenvalues with greater than \(\mu_0 > 0\);
   - a last eigenvalue \(\nu(b)\) such that \(\nu(0) = 0\).

3. \(D\nu(0)g(0, 0, 0) > 0\).
After a change of coordinates:

\[
\begin{align*}
\dot{x} &= A(x, y, z, c, \varepsilon)x, \\
\dot{y} &= B(x, y, z, c, \varepsilon)y, \\
\dot{z} &= h(z, c, \varepsilon)z + k(x, y, z, c, \varepsilon)xy, \\
\dot{c} &= \varepsilon((1, 0, \ldots, 0) + l(z, c, \varepsilon)z + L(x, y, z, c, \varepsilon)xy),
\end{align*}
\]

\[(x, y, z, c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}^{m-1},\]

- \(A(0, 0, 0, c, 0)\) has eigenvalues with negative real part,
- \(B(0, 0, 0, c, 0)\) has eigenvalues with positive real part,
- \(h(0, (0, c_2, \ldots, c_{m-1}), 0) = 0, \frac{\partial h}{\partial c_1} > 0.\)
Assume:

(1) $m = 2$ (for simplicity).

(2) For each $\varepsilon$, $M_\varepsilon$ is a submanifold of $xyzc$-space of dimension $l$.

(3) $M = \{(x, y, z, c, \varepsilon) : (x, y, z, c) \in M_\varepsilon\}$ is itself a manifold.

(4) $M_0$ meets $xzc$-space transversally at a point $(x_*, 0, \delta, c_*)$ with $\delta \neq 0$ and $c_* < 0$. We may assume that $M \subset \{(x, y, z, c, \varepsilon) : z = \delta\}$.

Each $M_\varepsilon$ meets $xzc$-space transversally at $(x, y, z, c) = (x(\varepsilon), 0, c(\varepsilon), \delta)$ with $(x(0), c(0)) = (x_*, c_*)$. 
Define Poincare maps on $z = \delta$ by $c \rightarrow \pi_\varepsilon(c)$.

Define $\pi_0$ implicitly by

$$\int_c^{\pi_0(c)} h(0, u, 0) \, du = 0.$$  

$\pi_\varepsilon \to \pi_0$, along with its derivatives, as $\varepsilon \to 0$ (De Maesschalck, 2008).

Under the forward flow, each $M_\varepsilon$ becomes a manifold $M_\varepsilon^*$ of dimension $l + 1$. 

\[\text{diagram showing the flow of } z = \delta \text{ with } c \rightarrow \pi_\varepsilon(c)\]
Theorem 4 (Liu’s Exchange Lemma, 2000). In $zc$-space, consider a short integral curve $C_\varepsilon$ through $(z, c) = (\delta, \pi_\varepsilon(c(\varepsilon)))$. Let

$$A_\varepsilon = \{(x, y, z, c): x = 0, \|y\| \text{ is small}, (z, c) \in C_\varepsilon\}.$$  

Then $M_\varepsilon^*$ is close to $A_\varepsilon$. As $\varepsilon \to 0$ the distance goes to 0 exponentially.
Gain-of-Stability Turning Points
(Rarefactions in the Dafermos Regularization)

Consider the system

\[
\begin{align*}
\dot{u} &= v, \\
\dot{v} &= (A(u) - xI)v, \\
\dot{x} &= \varepsilon,
\end{align*}
\]

with \((u, v, x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\) and \(A(u)\) an \(n \times n\) matrix.

Let \(n = k + l + 1\). Assume that on an open set \(U\) in \(\mathbb{R}^n\):

- There are numbers \(\lambda_1 < \lambda_2\) such that \(A(u)\) has
  - \(k\) eigenvalues with real part less than \(\lambda_1\),
  - \(l\) eigenvalues with real part greater than \(\lambda_2\),
  - a simple real eigenvalue \(\lambda(u)\) with \(\lambda_1 < \lambda(u) < \lambda_2\).
- \(A(u)\) has an eigenvector \(r(u)\) for the eigenvalue \(\lambda(u)\) such that \(D\lambda(u)r(u) = 1\).

Notice \(ux\)-space is invariant for every \(\varepsilon\). For \(\varepsilon = 0\) it consists of equilibria, but loses normal hyperbolicity along the surface \(x = \lambda(u)\).
Choose $u_\ast \in U, x_\ast, x^\ast$ such that $\lambda_1 < x_\ast < \lambda(u_\ast) < x^\ast < \lambda_2$. Let
\[ U_\ast = \{(u, v, x) : u \in U, v = 0, |x - x_\ast| < \delta\}, \]
\[ U^\ast = \{(u, v, x) : u \in U, v = 0, |x - x^\ast| < \delta\}. \]

For $\varepsilon = 0$, $U_\ast$ and $U^\ast$ are normally hyperbolic manifolds of equilibria of dimension $n + 1$. For $U_\ast$, the stable and unstable manifolds of each point have dimensions $k$ and $l + 1$ respectively; for $U^\ast$, the stable and unstable manifolds of each point have dimensions $k + 1$ and $l$ respectively.

For $\varepsilon > 0$, $U_\ast$ and $U^\ast$ are normally hyperbolic invariant manifolds on which the system reduces to $\dot{u} = 0, \dot{x} = \varepsilon$. 
For each $\varepsilon \geq 0$, let $M_\varepsilon$ be a submanifold of $uvx$-space of dimension $l + 1 + p$, $0 \leq p \leq n - 1$. Assume:

- $M = \{(u, v, x, \varepsilon) : (u, v, x) \in M_\varepsilon\}$ is itself a manifold.
- $M_0$ is transverse to $W^s_0(U_*)$ at a point in the stable fiber of $(u_*, 0, x_*)$. The intersection of $M_0$ and $W^s_0(U_*)$ is a smooth manifold $S_0$ of dimension $p$.
- $S_0$ projects smoothly to a submanifold $Q_0$ of $ux$-space of dimension $p$.
- The vector $(\dot{u}, \dot{x}) = (0, 1)$ is not tangent to $Q_0$. Therefore $Q_0$ projects smoothly to a submanifold $R_0$ of $u$-space of dimension $p$.
- $r(u_*)$ is not tangent to $R_0$.

Under the flow, each $M_\varepsilon$ becomes a manifold $M^*_\varepsilon$ of dimension $l + 2 + p$. 
Let $\phi(t, u)$ be the flow of $\dot{u} = r(u)$. Choose $t^* > 0$ such that $\lambda(u_*) + t^* < x^*$. Let $R_0^* = \bigcup_{|t-t^*|<\delta} \phi(t, R_0)$, $P_0^* = \{(u, v, x) : u \in R_0^*, v = 0, |x-x^*| < \delta\}$. $R_0^*$ and $P_0^*$ have dimensions $p + 1$ and $p + 2$ respectively.

Let $u^* = \phi(t^*, u_*)$.

**Theorem 7.** Near $(u^*, 0, x^*)$, $M_\varepsilon^*$ is close to $W_0^u(P_0^*)$. 
How the flow on the normally hyperbolic invariant manifold is analyzed

There is a normally hyperbolic invariant manifold with coordinates \((u, z_1, x, \varepsilon)\) with \(z_1\) a coordinate along \(r(u)\) in \(v\)-space.

The equilibria \(z_1 = \varepsilon = 0\) lose normal hyperbolicity when \(x = \lambda(u)\). We therefore make the change of variables \(x = \lambda(u) + \sigma\) and blow up the set \(z_1 = \sigma = \varepsilon = 0\):

\[
\begin{align*}
    u &= u, \\
    z_1 &= \bar{r}^2 \bar{z}_1, \\
    \sigma &= \bar{r} \bar{\sigma}, \\
    \varepsilon &= \bar{r}^2 \bar{\varepsilon},
\end{align*}
\]

with \(\bar{z}_1^2 + \bar{\sigma}^2 + \bar{\varepsilon}^2 = 1\).

For the new system, the spherical cylinder \(\bar{r} = 0\) consists entirely of equilibria. Divide by \(\bar{r}\) to desingularize.
Blown-up flow for fixed $u$. The $\varepsilon$-axis points toward you.
Generalized Deng’s Lemma

In the literature, there are three ways to prove exchange lemmas:

- Jones and Kopell’s approach, which is to follow the tangent space to $M_\epsilon$ forward using the extension of the linearized differential equation to differential forms.
- Brunovsky’s approach, which is to locate $M^*_\epsilon$ by solving a boundary value problem in Silnikov variables.

We follow Brunovsky’s approach, which is based on work of Bo Deng (1990). Brunovsky generalized a lemma of Deng that gives estimates on solutions of boundary value problems in Silnikov variables. Our proof of the Generalized Exchange Lemma is based on a further generalization of Deng’s Lemma.
Let \((x, y, c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m\). Let \(V\) be an open subset of \(\mathbb{R}^m\). On a neighborhood of \(\{0\} \times \{0\} \times V\), consider the \(C^{r+1}\) differential equation

\[
\begin{align*}
\dot{x} &= A(x, y, c)x, \\
\dot{y} &= B(x, y, c)y, \\
\dot{c} &= C(c) + E(x, y, c)xy.
\end{align*}
\]

Let \(\phi(t, c)\) be the flow of \(\dot{c} = C(c)\). For each \(c \in V\) there is a maximal interval \(I_c\) containing 0 such that \(\phi(t, c) \in V\) for all \(t \in I_c\). Let the linearized solution operator of the system, with \(\varepsilon = 0\), along the solution \((0, 0, \phi(t, c^0))\) be

\[
\begin{pmatrix}
\bar{x}(t) \\
\bar{y}(t) \\
\bar{c}(t)
\end{pmatrix} =
\begin{pmatrix}
\Phi^s(t, s, c^0) & 0 & 0 \\
0 & \Phi^u(t, s, c^0) & 0 \\
0 & 0 & \Phi^c(t, s, c^0)
\end{pmatrix}
\begin{pmatrix}
\bar{x}(s) \\
\bar{y}(s) \\
\bar{c}(s)
\end{pmatrix}
\]

Assume:

(E1) There are numbers \(\lambda_0 < 0 < \mu_0, \beta > 0, \) and \(M > 0\) such that for all \(c^0 \in N\) and \(s, t \in I_{c^0}\),

\[
\begin{align*}
\|\Phi^s(t, s, c^0)\| &\leq Me^{\lambda_0(t-s)} & \text{if } t \geq s, \\
\|\Phi^u(t, s, c^0)\| &\leq Me^{\mu_0(t-s)} & \text{if } t \leq s, \\
\|\Phi^c(t, s, c^0)\| &\leq Me^{\beta|t-s|} & \text{for all } t, s.
\end{align*}
\]

(E2) \(\lambda_0 + r\beta < 0 < \lambda_0 + \mu_0 - r\beta\).
We wish to study solutions of Silnikov’s boundary value problem on an interval 
\(0 \leq t \leq \tau\):
\[
x(0) = x^0, \quad y(\tau) = y^1, \quad c(0) = c^0.
\]

We denote the solution by \((x, y, c)(t, \tau, x^0, y^1, c^0)\).

**Theorem 9 (Generalized Deng’s Lemma, S. 2008).** Let \(V_0\) and \(V_1\) be compact subsets of \(V\) such that \(V_0 \subset \text{Int} \ V_1\). For each \(c^0 \in N_0\) let \(J_{c^0}\) be the maximal interval such that \(\phi(t, c^0) \in \text{Int} \ (V_1)\) for all \(t \in J_{c^0}\). Then for \(\lambda\) and \(\mu\) a little closer to 0 than \(\lambda_0\) and \(\mu_0\), there is a number \(\delta_0 > 0\) such that if \(\|x^0\| \leq \delta_0\), \(\|y^1\| \leq \delta_0\), \(c^0 \in N_0\), and \(\tau > 0\) is in \(J_{c^0}\), then Silnikov’s boundary value problem has a solution \((x, y, c)(t, \tau, x^0, y^1, c^0)\) on the interval \(0 \leq t \leq \tau\). Moreover, there is a number \(K > 0\) such that for all \((t, \tau, x^0, y^1, c^0)\) as above,
\[
\|x(t, \tau, x^0, y^1, c^0)\| \leq Ke^{\lambda t},
\]
\[
\|y(t, \tau, x^0, y^1, c^0)\| \leq Ke^{\mu(t-\tau)},
\]
\[
\|c(t, \tau, x^0, y^1, c^0) - \phi(t, c^0)\| \leq Ke^{\lambda t + \mu(t-\tau)}.
\]

In addition, if \(i\) is any \(|i|\)-tuple of integers between 1 and \(2 + k + l + m\), with \(1 \leq |i| \leq r\), then
\[
\|D_i x(t, \tau, x^0, y^1, c^0)\| \leq Ke^{(\lambda + |i|\beta)t},
\]
\[
\|D_i y(t, \tau, x^0, y^1, c^0)\| \leq Ke^{(\mu - |i|\beta)(t-\tau)},
\]
\[
\|D_i c(t, \tau, x^0, y^1, c^0) - D_i \phi(t, c^0)\| \leq Ke^{(\lambda + |i|\beta)t + (\mu - |i|\beta)(t-\tau)}.
\]