1. Given \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\). Write down the following:

   (a) the formula of Lagrange polynomial interpolation.
   
   (b) the Newton polynomial interpolation using the divided difference.
   
   (c) the error estimate.
   
   (d) Show that \( \sum_{i=0}^{n} l_i(x) = 1 \) if \( n > 0 \), where \( l_i(x) \) is the Lagrange polynomial.

Let the data be \((1, 0), (-1, 3), (0, -1)\), write down one of the two polynomial interpolations.

**Solution:**

(a) Lagrange polynomial interpolation:

\[
p_n(x) = \sum_{i=0}^{n} l_i(x)y_i, \quad l_i(x) = \prod_{j=0, j \neq i}^{n} \frac{x-x_j}{x_i-x_j}
\]

(b) Newton polynomial interpolation:

\[
p_n(x) = \sum_{i=0}^{n} f[x_0, x_1, \ldots, x_i] \omega_i(x), \quad \omega_i(x) = (x-x_0)(x-x_1)\cdots(x-x_{i-1})
\]

(c) We know that \( f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \omega_{n+1}(x) \).

(d) Take \( f(x) \equiv 1 \), then \( f^{(n+1)}(\xi) \equiv 0 \) when \( n > 0 \). That is, the interpolation is exact.

Therefore, we use the Lagrange polynomial interpolation to get

\[
1 \equiv f(x) = p_n(x) = \sum_{i=0}^{n} l_i(x).
\]

Using Newton divided difference, we get

\[
p_2(x) = -\frac{3}{2}(x-1) + \frac{5}{2}(x-1)(x+1).
\]

2. Find the divided differences:

   (a) \( f[0, 1, 2, \ldots, 7] \) if \( f(x) = x^7 + x^3 + 1 \).

   (b) \( f[0, 1, 2, \ldots, 7, 8] \) if \( f(x) = x^7 + x^3 + 1 \).

   (c) \( f[x_0, x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}, x_n] \) if \( f(x) = l_i(x) \).

**Solution:**

(a) Using \( f[x_0, x_1, \ldots, x_i] = \frac{f(i)}{i!} \), we get

\[
(a) \quad f[0, 1, 2, \ldots, 7] = \frac{7!}{7!} = 1
\]

\[
(b) \quad f[0, 1, 2, \ldots, 7, 8] = \frac{f(8)}{8!} = 0.
\]
(c) Using \( f[x_0, x_1, \cdots, x_i] = \sum_{j=0}^{i} \frac{f(x_j)}{\omega^{i+1}_{i+1}(x_j)} = 0 \) since \( l_i(x_j) = 0 \) if \( i \neq j \).

3. (a) Give the definition of a cubic spline \( S_3(x) \) given the nodal points \( x_0, x_1, \cdots, x_n \). What are the three common boundary conditions?

(b) What is a B-spline interpolation? List some properties of a B-spline interpolation.

Solution: (a) A cubic spline \( S_3(x) \) is piecewise cubic function between \( [a, b] \), \( S_3(x) \) has up to second order continuous derivatives. The three commonly used boundary conditions are: (1) Natural spline: \( S'_3(a) = S''_3(b) = 0 \). (2) Clamped spline: \( S'_3(a) = f'(a) \) and \( S'_3(b) = f'(b) \) are given. (3): Periodic spline: \( S^{(k)}_3(a) = S^{(k)}_3(b) \) for \( k = 0, 1, 2 \).

(b) A B-spline interpolation has the following form

\[
S_3(x) = \sum_{i=-3}^{n-1} \alpha_i N_i(x), \quad S_3(x_i) = f(x_i)
\]

where \( N_i(x) = g[x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}; t] \) with \( g(x, t) = (x-t)^3 \). \( 0 \leq N_i(x) \leq 1 \) and has local support.

Also if \( S_{3,f} \) is the cubic spline interpolation, then we have \( \int_a^b |S''_{3,f}(x)|^2 \, dx \leq \int_a^b |f''(x)|^2 \, dx \) and \( \int_a^b |f'' - S''_{3,f}(x)|^2 \, dx \leq \int_a^b |f''(x) - S''_3(x)|^2 \, dx \) for any spline function \( S''_3(x) \).

4. Write down the Newton-Cotes quadrature closed, and open formulas and give an error estimate.

Solution:

\[
\int_a^b f(x) \, dx \approx h \sum_{i=0}^{n} w_i f(x_i)
\]

\[
w_i = \int_0^{n+1} \prod_{k=0, k \neq i}^n \frac{t-k}{i-k} \, dt \quad \text{for closed formula with } h = \frac{b-a}{n}
\]

\[
w_i = \int_{-1}^{n+1} \prod_{k=0, k \neq i}^n \frac{t-k}{i-k} \, dt \quad \text{for open formula with } h = \frac{b-a}{n+2}
\]

The error estimate is

\[
E_n(f) = \int_a^b f(x) \, dx - I_n(f) = \begin{cases} 
\frac{M_n}{(n+2)!} h^{n+3} f^{(n+2)}(\xi) & \text{if } n \text{ is even.} \\
\frac{K_n}{(n+1)!} h^{n+2} f^{(n+1)}(\xi) & \text{if } n \text{ is odd.}
\end{cases}
\]

5. (a) Write down (a) mid-point rule; (b) trapezoidal rule; (c) Simpson rule to evaluate the integral \( \int_{x_i}^{x_i+h} x^3 \, dx \) and find the error estimates in terms of \( h \).
(b) Find out and show the algebraic precision of the trapezoidal rule.

Solution:

\[
\begin{align*}
\int_{x_i}^{x_i+h} x^3 \, dx &\approx h (x_i + h)^3 \\
\int_{x_i}^{x_i+h} x^3 \, dx &\approx \frac{h}{2} \left( x_i^3 + (x_i + h)^3 \right) \\
\int_{x_i}^{x_i+h} x^3 \, dx &\approx \frac{h}{6} \left( x_i^3 + 4(x_i + h/2)^3 + (x_i + h)^3 \right)
\end{align*}
\]

The error estimates for the mid-point rule and the trapezoidal rule are both \(O(h^3)\) and for the Simpson rule, the error is 0. It is easy to check that \(\int_a^b x^k \, dx = \frac{b^k - a^k}{k+1}\) for \(k = 0, 1\), but \(\int_a^b x^2 \neq \frac{b^2 - a^2}{2} \left( a^2 + b^2 \right)\), therefore the algebraic precision of the trapezoidal rule is 1. The algebraic precision of the mid-point rule and the Simpson rule are 1 and 3, respectively.

6. Set up a system of equations for the coefficients \(\alpha_1, \alpha_2, x_1\) and \(x_2\) so that the following quadrature

\[
\int_{-1}^{1} f(x) \, dx \approx \alpha_1 f(x_1) + \alpha_2 f(x_2)
\]

has as high algebraic precision as possible. What is likely the algebraic precision? Note: You do not need to solve the system of equation.

Solution: Since there are 4 parameters, we can set 4 equations

\[
\begin{align*}
x^0 : \quad & \int_{-1}^{1} dx = 2 = \alpha_1 + \alpha_2 \\
x^1 : \quad & \int_{-1}^{1} x \, dx = 0 = \alpha_1 x_1 + \alpha_2 x_2 \\
x^2 : \quad & \int_{-1}^{1} x^2 \, dx = \frac{2}{3} = \alpha_1 x_1^2 + \alpha_2 x_2^2 \\
x^3 : \quad & \int_{-1}^{1} x^3 \, dx = 0 = \alpha_1 x_1^3 + \alpha_2 x_2^3
\end{align*}
\]

The algebraic precision of the quadrature formula is likely to be 3.

7. If \(A(h) = A_0 + Ch^3 + O(h^5)\), find the formula of the Richardson extrapolation.

Solution:

\[
B_h = \frac{A(\delta h) - \delta^3 A(h)}{1 - \delta^3}
\]

\[
B_h = \frac{A(h/2) - A(h)}{1 - (\frac{1}{2})^3} = \frac{8A(h/2) - A(h)}{7}, \quad \delta = \frac{1}{2}.
\]