Exercises 2.2

In Exercises 1–4, a 2π-periodic function is specified on the interval [− π, π].
(a) Plot the function on the interval [−3π, 3π].
(b) Plot its Fourier series (without computing it) on the interval [−3π, 3π].

1.

2.

3.

4.

For the Fourier series of the functions in Exercises 1–4, see Exercises 19, 21, and 22.

In Exercises 5–16, the equation of a 2π-periodic function is given on an interval of length 2π. You are also given the Fourier series of the function. (a) Derive the given Fourier series.
(b) Plot the function and the Nth partial sums of its Fourier series for N = 1, 2, ⋯, 20. Discuss the convergence of the partial sums by considering their graphs. Be specific at the points of discontinuity.

5. \( f(x) = |x| \) if \( −π \leq x < π \). Fourier series: \[ \frac{π}{2} \sum_{k=0}^{∞} \frac{1}{(2k+1)^2} \cos(2k+1)x. \]

6. \( f(x) = \begin{cases} 1 & \text{if } 0 < x < π/2, \\ -1 & \text{if } -π/2 < x < 0, \\ 0 & \text{if } π/2 < |x| < π. \end{cases} \)

Fourier series: \[ \frac{2}{π} \sum_{n=1}^{∞} \frac{1}{n} \left(1 - \cos \frac{πn}{2}\right) \sin nx. \]

7. \( f(x) = |\sin x| \) if \( −π \leq x \leq π \). Fourier series: \[ \frac{2}{π} \sum_{k=1}^{∞} \frac{1}{(2k)^2 - 1} \cos 2kx. \]

8. \( f(x) = |\cos x| \) if \( −π \leq x \leq π \).

Fourier series: \[ \frac{2}{π} - \frac{4}{π} \sum_{k=1}^{∞} \frac{(-1)^k}{(2k)^2 - 1} \cos 2kx. \]

[Hint: You can compute directly, or, if you have done Exercise 7, substitute \( x - π/2 \) for \( x \).]

9. \( f(x) = x^2 \) if \( −π \leq x \leq π \). Fourier series: \[ \frac{π^2}{3} + 4 \sum_{n=1}^{∞} \frac{(-1)^n}{n^2} \cos nx. \]

10. \( f(x) = 1 - \sin x + 3 \cos 2x \). Fourier series: same as \( f(x) \).

11. \( f(x) = \sin^2 x; \quad f(x) = \cos^2 x \).

Fourier series: \[ \frac{1}{2} - \frac{\cos 2x}{4}; \quad \frac{1}{2} + \frac{\cos 2x}{4}. \]
12. \( f(x) = \pi^2 x - x^3 \) if \(-\pi < x < \pi\).

Fourier series: \[ 12 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \sin n x. \]

13. \( f(x) = x \) if \(-\pi < x < \pi\).

Fourier series: \[ 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n x. \]

[Hint: Let \( x = \pi - t \) in Example 1.]

14. For parts (a) and (b), take \( c, d > 0 \) and \( d < \pi \). For part (c), take \( c = d = \pi/2 \).

\[ f(x) = \begin{cases} 
0 & \text{if } -\pi \leq x \leq -d, \\
\frac{x}{d} (x + d) & \text{if } -d \leq x \leq 0, \\
-\frac{x}{d} (x - d) & \text{if } 0 \leq x \leq d, \\
0 & \text{if } d \leq x \leq \pi.
\end{cases} \]

Fourier series: \[ \frac{cd}{2\pi} + \frac{4c}{d\pi} \sum_{n=1}^{\infty} \frac{\sin^2\left(\frac{d\pi}{n}\right)}{n^2} \cos nx. \]

15. \( f(x) = e^{-|x|} \) if \(-\pi < x \leq \pi\).

Fourier series: \[ \frac{e^\pi - 1}{\pi e^\pi} + \frac{2}{\pi e^\pi} \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} (e^\pi - (-1)^n) \cos nx. \]

16. \( f(x) = \begin{cases} 
1/(2c) & \text{if } |x - d| < c, \\
0 & \text{if } c < |x - d| < \pi,
\end{cases} \]

where \( 0 < c < \pi \) and \( d \) is arbitrary.

Fourier series: \[ \frac{1}{2\pi} + \frac{1}{c\pi} \sum_{n=1}^{\infty} \left( \frac{\sin(nc) \cos(nd)}{n} \cos nx + \frac{\sin(nc) \sin(nd)}{n} \sin nx \right). \]

For part (c), take \( c = d = \pi/4 \).

17. Use the Fourier series of Exercise 9 to obtain

\[ \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots. \]

18. Use the Fourier series of Exercise 13 to obtain

\[ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots. \]

19. Derive the Fourier series of the given function as indicated, without computing its Fourier coefficients.

(a) The function in Exercise 1, by using the Fourier series of the square wave in
is and

Figure 10 illustrates the convergence of the Fourier series to $f(x)$. Along with $f(x)$, we have plotted the partial sums $s_2(x)$ and $s_4(x)$. The graphs of $s_4(x)$ and $f(x)$ can hardly be distinguished from one another, which suggests that the Fourier series converges very fast to $f(x)$. 

In the next section we use Fourier series of even and odd functions to periodically extend functions that are defined on finite intervals. As we will see in Chapter 3, this process will be needed in solving partial differential equations by means of Fourier series.

**Exercises 2.3**

In Exercises 1–10, a $2p$-periodic function is given on an interval of length $2p$. (a) State whether the function is even, odd, or neither. (b) Derive the given Fourier series, and determine its values at the points of discontinuity. (Most of these Fourier series can be derived from earlier examples and exercises, as illustrated by Examples 2 and 3.)

1. 

$$f(x) = \begin{cases} 
1 & \text{if } 0 < x < p, \\
-1 & \text{if } -p < x < 0.
\end{cases}$$

Fourier series: 

$$f(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin \left( \frac{(2k+1)\pi}{p}x \right).$$

2. $f(x) = x$ if $-p < x < p$. [Hint: Exercise 13, Section 2.2.]

Fourier series: 

$$f(x) = \frac{2p}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \left( \frac{n\pi}{p}x \right).$$

3. $f(x) = a \left( 1 - \left( \frac{x}{p} \right)^2 \right)$ if $-p \leq x \leq p$, ($a \neq 0$).

Fourier series: 

$$f(x) = \frac{2}{3}a + 4a \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n\pi)^2} \cos \left( \frac{n\pi}{p}x \right).$$

4. $f(x) = x^2$ if $-p \leq x \leq p$. [Hint: Use Exercise 3.]

Fourier series: 

$$f(x) = \frac{p^2}{3} - \frac{4p^2}{\pi^2} \left[ \cos \left( \frac{\pi}{p}x \right) - \frac{1}{2^2} \cos \left( \frac{2\pi}{p}x \right) + \frac{1}{3^2} \cos \left( \frac{3\pi}{p}x \right) - \cdots \right].$$
It is a remarkable fact that the cosine series and the sine series have the same values on the intervals \((0, 1), (2, 3), (-2, -1), \ldots\). ■

**EXAMPLE 2** Half-range expansions

Consider the function \(f(x) = \sin x\), \(0 \leq x \leq \pi\). If we take its odd extension, we get the usual sine function, \(f_2(x) = \sin x\) for all \(x\). Thus, the sine series expansion is just \(\sin x\).

**Figure 3** (a) \(f(x) = \sin x, 0 \leq x \leq \pi\). (b) Odd extension of \(f, \sin x\). (c) Even extension of \(f, |\sin x|\).

If we take the even extension of \(f\), we get the function \(|\sin x|\). The Fourier series of this even function can be obtained from Exercise 7, Section 2.2. Thus the cosine series (of \(\sin x\)) is

\[
\sin x = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k)^2 - 1} \cos 2kx, \quad 0 \leq x \leq \pi.
\]

■

**Exercises 2.4**

In Exercises 1–8, (a) find the half-range expansions of the given function. (Use as much as possible series that you have encountered earlier.)

(b) To illustrate the convergence of the cosine and sine series, plot several partial sums of each and comment on the graphs.

1. \(f(x) = 1\) if \(0 < x < 1\).
2. \(f(x) = \pi - x\) if \(0 \leq x \leq \pi\).
3. \(f(x) = x^2\) if \(0 < x < 1\).
4. \(f(x) = \begin{cases} 
0 & \text{if } 0 \leq x < 1, \\
 x - 1 & \text{if } 1 \leq x < 2.
\end{cases}\)

5. \(f(x) = \begin{cases} 
1 & \text{if } a < x < b, \\
0 & \text{if } 0 < x < a \\
or b < x < p,
\end{cases}\)

where \(0 < a < b < p < \infty\). For (b), take \(p = 1, a = \frac{1}{4}, b = \frac{1}{2}\).

6. \(f(x) = \cos x\) if \(0 < x < \pi\).
7. \(f(x) = \cos x\) if \(0 \leq x \leq \frac{\pi}{2}\).
8. \(f(x) = x \sin x\) if \(0 < x < \pi\).
represents the average of \( f \) on the interval \([a, b]\). [Hint: Approximate the definite integral by a Riemann sum. Interpret the sum as an average, using the fact that the average of \( n \) numbers \( y_1, y_2, \ldots, y_n \) is \( \frac{y_1 + y_2 + \cdots + y_n}{n} \).]

The **Riemann zeta function** is defined for all \( s > 1 \) by

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.
\]

This function is very important in number theory and other branches of mathematics. We saw in Example 2 that \( \zeta(2) = \frac{\pi^2}{6} \). Using Parseval’s identity and various Fourier series expansions, you can compute \( \zeta(2k) \) for any positive integer \( k \). The following table contains some values that you are asked to derive in Exercises 7–11, below.

<table>
<thead>
<tr>
<th>( s )</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \zeta(s) ) = ( \sum_{n=1}^{\infty} \frac{1}{n^s} )</td>
<td>( \frac{\pi^2}{6} )</td>
<td>( \frac{\pi^4}{90} )</td>
<td>( \frac{\pi^6}{945} )</td>
<td>( \frac{\pi^8}{9450} )</td>
<td>( \frac{\pi^{10}}{93555} )</td>
</tr>
</tbody>
</table>

### 7. (a) Use Parseval’s identity and the Fourier series expansion

\[
\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad -\pi < x < \pi,
\]

to obtain

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

(b) From (a) obtain that \( \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{24} \).

(c) Combine (a) and (b) to derive the identity

\[
\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.
\]

### 8. Use the Fourier series expansion

\[
x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx, \quad -\pi < x < \pi,
\]

(Exercise 4, Section 2.3) and Parseval’s identity to calculate \( \zeta(4) \).

### 9. Use the Fourier series expansion in Exercise 12, Section 2.2, and Parseval’s identity to calculate \( \zeta(6) \).

### 10. The Bernoulli numbers \( B_n \) arise in many contexts in mathematics. They can be generated from the power series expansion

\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}.
\]
Section 2.9 Uniform Convergence and Fourier Series

With $M_k = \frac{1}{k^2}$. We thus infer from Theorem 5 that $u'(x) = \sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$. Thus the series can be differentiated term by term. 

In the next section we derive a powerful test for uniform convergence known as the Dirichlet test. Several examples are also presented, including interesting uniformly convergent series that cannot be differentiated term by term.

**Exercises 2.9**

1. $f_n(x) = \frac{\sin nx}{\sqrt{n}}; 0 \leq x \leq 2\pi$.
2. $f_n(x) = \frac{x}{1 + nx^2}; -1 \leq x \leq 1$.
3. $f_n(x) = \frac{n^2x}{1 + n^2x^2}; -1 \leq x \leq 1$.
4. $f_n(x) = e^{-nx}; 0 \leq x \leq 1$.
5. $f_n(x) = nx e^{-nx}; 0 \leq x \leq 1$.
6. $f_n(x) = e^{-n^2x^2} - e^{-2nx}; 0 \leq x \leq 1$.
7. $f_n(x) = \frac{nx}{n^2x^2 + 1}; 0 \leq x$.
8. $f_n(x) = \cos\left( \frac{x}{n} \right); 0 \leq x \leq \pi$.

In Exercises 9–18, use the Weierstrass M-test to establish the uniform convergence of the given series, on the given interval.

9. $\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}; \text{ all } x$.
10. $\sum_{k=1}^{\infty} \frac{(\cos kx + \sin kx)}{k^3}; \text{ all } x$.
11. $\sum_{k=0}^{\infty} \frac{2^k}{k!}; |x| \leq 10$.
12. $\sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{k!}; |z| \leq 700$.
13. $\sum_{k=0}^{\infty} (10x)^k; |x| \leq \frac{1}{11}$.
14. $\sum_{k=1}^{\infty} \frac{x}{10^k}; |x| \leq 9$.
15. $\sum_{k=0}^{\infty} x^k; |x| < .99$.
16. $\sum_{k=1}^{\infty} \frac{1}{x^2 + k^2}; \text{ all } x$.
17. $\sum_{k=1}^{\infty} \frac{(-1)^k}{|x| + k^2}; \text{ all } x$.
18. $\sum_{k=1}^{\infty} \frac{\cos(x/k)}{k^2}; \text{ all } x$.

19. Of the functions in Exercises 1–4, Section 2.2, which ones have a uniformly convergent Fourier series? Justify your answers without looking at the Fourier series.

20. The Fourier coefficients of a $2\pi$-periodic function are as follows: $a_0 = 0$, $a_n = \frac{(-1)^n}{n^2}$, and $b_n = \frac{1}{n}$, for all $n \geq 1$. Is the function continuous? Justify your answer.

21. The Fourier coefficients of a $2\pi$-periodic function are as follows: $a_0 = 1$, $a_n = \frac{1}{1+n^2}$, and $b_n = \frac{1}{n^2}$, for all $n \geq 1$. Is the function continuous? Justify your answer.

22. Give an example of a $2\pi$-periodic function $f(x)$ such that $f(x)$ is not continuous for all $x$ but $f^2(x)$ is continuous for all $x$.

23. Give an example of a $2\pi$-periodic function $f(x)$ such that the Fourier series of $f(x)$ is not uniformly convergent for all $x$ but the Fourier series of $f^2(x)$ is uniformly