Units and Dimensional Analysis

- Math modeling is used to solve real world problems. Most of quantities in the real world have units. Or physical quantities are measured using units.

- A unit of measurement is a definite magnitude of a physical quantity, defined and adopted by convention or by law, that is used as a standard for measurement of the same physical quantity. Any other value of the physical quantity can be expressed as a simple multiple of the unit of measurement.
Dimensional Analysis

- Purposes:
  - Check the correctness of mathematical models for physical problems by checking dimensional unit
  - Derive mathematical models
  - Reduce parameters through Non-dimensionalization process or (scaling)
  - Identify key parameters such as Reynolds number, peck number etc.
Units Examples

- **Length** is a physical quantity. The **meter** is a unit of length that represents a definite predetermined length. When we say 10 meters (or 10 m), we actually mean 10 times the definite pre-determined length called “meter (metre)".

- The definition, agreement, and practical use of units of measurement have played a crucial role in human endeavour from early ages up to this day. Different systems of units used to be very common. Now there is a global standard, the **International System of Units (SI)**, the modern form of the **metric system**. But English system is still in use (UK, US, some counties in British Commonwealth Union)
Units

The recommended scientific system of units is the SI system, which includes 7 basic units.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Unit</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>metre</td>
<td>m</td>
</tr>
<tr>
<td>Mass</td>
<td>kilogram</td>
<td>kg</td>
</tr>
<tr>
<td>Time</td>
<td>second</td>
<td>s</td>
</tr>
<tr>
<td>Electric current</td>
<td>ampere</td>
<td>A</td>
</tr>
<tr>
<td>Temperature</td>
<td>kelvin</td>
<td>K</td>
</tr>
<tr>
<td>Luminous intensity</td>
<td>candela</td>
<td>cd</td>
</tr>
<tr>
<td>Amount of substance</td>
<td>mole</td>
<td>mol</td>
</tr>
</tbody>
</table>
Large and small units

There is a wide range of sizes for all sorts of quantities. So we use special symbols for large and small multiples of the basic units.

<table>
<thead>
<tr>
<th>Multiplication factor</th>
<th>Prefix</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{12}$</td>
<td>tera</td>
<td>T</td>
</tr>
<tr>
<td>$10^9$</td>
<td>giga</td>
<td>G</td>
</tr>
<tr>
<td>$10^6$</td>
<td>mega</td>
<td>M</td>
</tr>
<tr>
<td>$10^3$</td>
<td>kilo</td>
<td>k</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>centi</td>
<td>c</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>milli</td>
<td>m</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>micro</td>
<td>μ</td>
</tr>
<tr>
<td>$10^{-9}$</td>
<td>nano</td>
<td>n</td>
</tr>
</tbody>
</table>
Commonly used units

There are also other commonly used units which are combinations of some of the SI units.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Unit</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force</td>
<td>newton</td>
<td>N (kg m s(^{-2}))</td>
</tr>
<tr>
<td>Energy</td>
<td>joule</td>
<td>J (kg m(^2) s(^{-2}))</td>
</tr>
<tr>
<td>Power</td>
<td>watt</td>
<td>W (J s(^{-1}) or kg m(^2) s(^{-3}))</td>
</tr>
<tr>
<td>Frequency</td>
<td>hertz</td>
<td>Hz (s(^{-1}))</td>
</tr>
<tr>
<td>Pressure</td>
<td>pascal</td>
<td>Pa (N m(^{-2}) or kg m(^{-1}) s(^{-2}))</td>
</tr>
</tbody>
</table>
Non-SI units

There are a number of non-SI units which are in common use by scientists and engineers.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Unit</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area</td>
<td>hectare</td>
<td>ha ( (= 10^4 \text{ m}^2))</td>
</tr>
<tr>
<td>Volume</td>
<td>litre</td>
<td>1 ( (= 10^{-3} \text{ m}^3))</td>
</tr>
<tr>
<td>Volume</td>
<td>millilitre</td>
<td>ml ( (= 10^{-6} \text{ m}^3))</td>
</tr>
<tr>
<td>Temperature</td>
<td>degree Celsius</td>
<td>°C ((0°C \approx 273 \text{ K}))</td>
</tr>
<tr>
<td>Mass</td>
<td>gram</td>
<td>g ( (= 10^{-3} \text{ kg}))</td>
</tr>
<tr>
<td>Mass</td>
<td>tonne</td>
<td>t ( (= 10^3 \text{ kg}))</td>
</tr>
<tr>
<td>Energy</td>
<td>kilowatt hour</td>
<td>kW h ((= 3.6 \times 10^6 \text{ J}))</td>
</tr>
<tr>
<td>Energy</td>
<td>electronvolt</td>
<td>eV ((\approx 1.6 \times 10^{-19} \text{ J}))</td>
</tr>
<tr>
<td>Energy</td>
<td>calorie</td>
<td>cal ((= 4.1868 \text{ J}))</td>
</tr>
<tr>
<td>Pressure</td>
<td>bar</td>
<td>bar ((= 10^5 \text{ Pa}))</td>
</tr>
<tr>
<td>Pressure</td>
<td>atmosphere</td>
<td>atm ((\approx 1.013 \times 10^5 \text{ Pa}))</td>
</tr>
</tbody>
</table>
Units Converting

The Fahrenheit scale of temperature

To convert from °F to °C,

$$°C = 5(°F - 32)/9 \sim (°F - 32)/2$$

To convert from °C to °F

$$°F = 9 °C /5 + 32 \sim 2 °C +32$$

Converting any units (almost anything)

http://www.onlineconversion.com/
Another system of units which is still in use is the British system.

<table>
<thead>
<tr>
<th>fps unit</th>
<th>SI equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>inch (in)</td>
<td>0.0254 m</td>
</tr>
<tr>
<td>foot (ft)</td>
<td>0.3048 m</td>
</tr>
<tr>
<td>mile (5280 ft) (5 miles ≈ 8 km)</td>
<td>1.609344 × 10^3 m</td>
</tr>
<tr>
<td>nautical mile (6080 ft)</td>
<td>1.853184 × 10^3 m</td>
</tr>
<tr>
<td>acre</td>
<td>4.046856 × 10^3 m^2 (≈ 0.4 ha)</td>
</tr>
<tr>
<td>pint (pt)</td>
<td>5.682613 × 10^{-4} m^3</td>
</tr>
<tr>
<td>gallon (gal)</td>
<td>4.54609 × 10^{-3} m^3</td>
</tr>
<tr>
<td>ounce (oz)</td>
<td>2.834952 × 10^{-2} kg</td>
</tr>
<tr>
<td>pound (lb)</td>
<td>0.45359237 kg</td>
</tr>
<tr>
<td>horsepower (hp)</td>
<td>7.457 × 10^2 W</td>
</tr>
</tbody>
</table>
An example

The price of milk in the UK is about 1.65 pounds every 6 pints. That in China is 33 RMB every 6 litre. Assume that 1 pound = 15 RMB.

Which is cheaper? (using the same unit)

2008: UK: 1.65/6 → 15*1.65/3.409=7.26, CH: 33/6=5.5
2013: UK: 1.65/6 → 9.37*1.65/3.409=4.53, CH: 33/6=5.5

There is a 50% price rise in China recently.
Which is cheaper? CH: 33*1.5/6=5.5*1.5=8.25
Dimensional Analysis

- A model which describes a physical, biological, economic or managerial system involves a variety of parameters or variables.
- With each variable or parameter we can associate a dimension.

- area = length$^2$
- velocity = length / time
Dimensions: Definition

- All *mechanical* quantities can be expressed in terms of the fundamental quantities:
  - *mass (M) or kg, length (L) or m, time (T) or s*
- Other physical quantities can be expressed as a combination of these 3 terms.
- The resultant combination is called the ‘*dimensions*’ of that physical quantity.
Dimensions: Definition

- We use square brackets [ ] to denote “the dimension of”

  - \([\text{area}] = L^2\)
  - \([\text{density}] = M \, L^{-3}\)
  - \([\text{force}] = M \, L \, T^{-2}\)
  - \([\text{speed}] = L \, T^{-1}\)
  - \([\text{angle}] = L \, L^{-1} = L^0\)
  - \([\text{weight}] = M \, L \, T^{-2}\)

Note, dimensions are independent of the units used.
## Full Dimensional List

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>M</td>
</tr>
<tr>
<td>Length</td>
<td>L</td>
</tr>
<tr>
<td>Time</td>
<td>T</td>
</tr>
<tr>
<td>Electric Charge</td>
<td>Q</td>
</tr>
<tr>
<td>Temperature</td>
<td>θ</td>
</tr>
<tr>
<td>Number of Moles</td>
<td>MOL</td>
</tr>
<tr>
<td>Luminosity</td>
<td>?</td>
</tr>
</tbody>
</table>
Dimensional Analysis

- Any sensible equation must be dimensionally consistent

\[ \text{[left-hand side]} = \text{[right-hand side]} \]

- It is a good idea to carry out this check on all the equations appearing in a model

- This reveals any modeling errors
Dimensional Analysis

- Addition of terms only makes sense if each term has the same dimensions.
- For a proposed equation, each term must be checked for consistency.

\[ A = B + (C \times D) \]

- \( A \), \( B \) and \((C \times D)\) must have the same dimensions.
Determine the units for constants

Any constants appearing in equations can be

- Either be dimensionless (pure numbers)
- Or can have dimensions
Example

Suppose that we are modeling the force on a moving object due to air resistance. If we assume the magnitude of the force $F$ is proportional to the square of the speed $v$:

$$F = kv^2$$

In dimensions: $[F] = [kv^2]$

$$MLT^{-2} = [k][LT^{-1}]^2 = [k]L^2T^{-2}$$

* For consistency, we require $[k] = ML^{-1}$
* $k$ is measured in kg m$^{-1}$. 
Dimensional Analysis

- If expressions involving \( \exp(at) \) or \( \sin(at) \) appear in our model, where \( t \) stands for time.
- The parameter \( a \) must have dimensions \( T^{-1} \) so that \( at \) is a dimensionless number.
- If an equation involves a derivative, the dimensions of the derivative are given by the ratio of the dimensions.
- If \( p \) is the pressure in a fluid at any point, \( z \) is the depth, then
Dimensional Analysis: example

\[
\begin{align*}
\left[ \frac{dp}{dz} \right] &= \left[ p \right] = \frac{ML^{-1}T^{-2}}{L} = ML^{-2}T^{-2} \\
\left[ \frac{\partial p}{\partial t} \right] &= \left[ p \right] = \frac{ML^{-1}T^{-2}}{T} = ML^{-2}T^{-3} \\
\left[ \frac{\partial^2 v}{\partial x^2} \right] &= \left[ v \right] = \frac{L}{L^2} = L^{-1}T^{-1}
\end{align*}
\]
Dimensional Analysis: Pendulum

Suppose that we are trying to develop a model which will predict the period $t$ of a swinging pendulum:

List of Factors  (attention restricted to 4 factors)

- the length $l$
- the mass $m$
- the angle $\theta$
- acceleration $g$ due to gravity

• Assume that the period

$$[t] = [kl^a m^b g^c \theta^d]$$

where $a, b, c, d$ and $k$ are unknown real numbers.
Dimensional Analysis: Pendulum

Considering dimensions, we have

\[ [t] = [kl^a m^b g^c \theta^d] \]

\( \theta \) is dimensionless and \( k \) is assumed to be as well, so

\[ T = L^a M^b (LT^{-2})^c \]

Equating powers of \( M, L \) and \( T \) on both sides

\[
\begin{align*}
L : 0 &= a + c \\
M : 0 &= b \\
T : 1 &= -2c
\end{align*}
\]

\[ \Rightarrow b = 0, \quad c = -\frac{1}{2}, \quad a = -c = \frac{1}{2} \]
Dimensional Analysis: Pendulum

\[ [t] = [kl^a m^b g^c \theta^d] \]

therefore

\[ t = kl^{1/2} g^{-1/2} \theta^d \]

At present, \( d \) is unresolved, i.e., it can take any value

Summing terms of this form leads to the general result

\[ t = f(\theta) l^{1/2} g^{-1/2} \]
Dimensional Analysis: Fluid

Example: The pressure $p$ at a depth $h$ below the surface of a fluid of density $\rho$ is given by $p = \rho gh$, where $g$ is the acceleration due to gravity. Check the dimensions.

$$[p] = \left[\frac{\text{force}}{\text{area}}\right] = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$$

$$[\rho] = ML^{-3}$$

$$[g] = [\text{acceleration}] = LT^{-2}$$

$$[h] = L$$

$$[\rho gh] = ML^{-3}LT^{-2}L = ML^{-1}T^{-2}$$

The dimensions are consistent.
Nondimensionalization

- Nondimensionalization is the partial or full removal of units from an equation involving physical quantities by a suitable substitution of variables. This technique can simplify and parameterize problems where measured units are involved. It is closely related to dimensional analysis. In some physical systems, the term scaling is used interchangeably with nondimensionalization, in order to suggest that certain quantities are better measured relative to some appropriate unit. These units refer to quantities intrinsic to the system, rather than units such as SI units.

- Nondimensionalization can also recover characteristic properties of a system. The technique is especially useful for systems that can be described by differential equations.

- Nondimensionalization can reduce the number of parameters and keep the most important ones such as Reynolds number.
Nondimensionalization steps

To nondimensionalize a system of equations, one must do the following:

1. Identify all the independent and dependent variables;
2. Replace each of them with a quantity scaled relative to a characteristic unit of measure to be determined;
3. Divide through by the coefficient of the highest order polynomial or derivative term;
4. Choose judiciously the definition of the characteristic unit for each variable so that the coefficients of as many terms as possible become 1;
5. Rewrite the system of equations in terms of their new dimensionless quantities.
An illustrative example

\[ a \frac{dx}{dt} + bx = Af(t). \]

1. In this equation the independent variable here is \( t \), and the dependent variable is \( x \).

2. Set \( x = \chi x_c, \ t = \tau t_c \). This results in the equation

\[ a \frac{x_c}{t_c} \frac{d\chi}{d\tau} + bx_c \chi = Af(\tau t_c) \overset{\text{def}}{=} AF(\tau). \]

3. The coefficient of the highest ordered term is in front of the first derivative term. Dividing by this gives

\[ \frac{d\chi}{d\tau} + \frac{bt_c}{a} \chi = \frac{At_c}{ax_c} F(\tau). \]

4. The coefficient in front of \( \chi \) only contains one characteristic variable \( t_c \), hence it is easiest to choose to set this to unity first:

\[ \frac{bt_c}{a} = 1 \Rightarrow t_c = \frac{a}{b}, \text{ Subsequently, } \frac{At_c}{ax_c} = \frac{A}{bx_c} = 1 \Rightarrow x_c = \frac{A}{b}. \]

5. The final dimensionless equation in this case becomes completely independent of any parameters with units:

\[ \frac{d\chi}{d\tau} + \chi = F(\tau). \]
Dimensional Analysis

- Well known to any engineer
- This simple information is also remarkably useful in mathematical modelling
- Will consider ideas in context of the modelling of physical systems, but are easily extended to any application
Dimensional Analysis

- Check the validity of equations proposed during the modelling process
- Find a number of independent parameter groups (and calculate them)
- Find the relative sizes of various terms when model equations are proposed
- Normalise problems in terms of non-dimensional variables whose typical scale is of the order of one, and hence simplify them