NOTES ON LINEAR NON-AUTONOMOUS SYSTEMS

XIAO-BIAO LIN

1. GENERAL LINEAR SYSTEMS

Consider the linear nonhomogeneous system

\[ y' = A(t)y + g(t) \]  

where \( A(t) \) and \( g(t) \) are continuous on an interval \( I \).

**Theorem 1.1.** If \( A(t) \), \( g(t) \) are continuous on some interval \( a \leq t \leq b \), if \( a \leq t_0 \leq b \), and if \( \eta \in \mathbb{R}^n \), then the system (1.1) has a unique solution \( \phi(t) \) satisfying the initial condition \( \phi(t_0) = \eta \) and existing on the interval \( a \leq t \leq b \).

1.1. Linear Homogeneous Systems. Consider the linear homogeneous system associated with (1.1)

\[ y' = A(t)y. \]

For a homogeneous system, \( \phi(t) = 0 \) is the only solution that satisfies \( \phi(t_0) = 0 \). Moreover, if \( \phi_1 \) and \( \phi_2 \) are any solutions of (1.2) on an interval \( I \), and \( c_1 \) and \( c_2 \) are any constants, then \( c_1\phi_1 + c_2\phi_2 \) is again a solution of (1.2).

**Definition 1.1.** A set of vectors \( v_1, v_2, \ldots, v_k \) is linearly dependent if there exist scalers \( c_1, c_2, \ldots, c_k \), not all zero, such that the linear combination

\[ c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0. \]

A set of vectors \( v_1, v_2, \ldots, v_k \) is linearly independent if it is not linearly dependent.

A set \( S \) of vectors is said to form a basis of a vector space \( V \) if it is linearly independent and if every vector in \( V \) can be expressed as a linear combination of vectors in \( S \).

We can define the dimension of a particular vector space \( V \) to be the number of elements in any basis of \( V \). A vector space is called finite-dimensional if is has a finite basis.

**Theorem 1.2.** If the complex \( n \times n \) matrix \( A(t) \) is continuous on an interval \( I \), then the solutions of the system (1.2) on \( I \) form a vector space of dimension \( n \) over the complex numbers.

We say that the linearly independent solutions \( \phi_1, \phi_2, \ldots, \phi_n \) form a fundamental set of solutions. There are infinitely many different fundamental sets of solutions of (1.2).

A matrix of \( n \) rows whose columns are solutions (1.2) is called a solution matrix. An \( n \times n \) solution matrix whose columns form a fundamental set of solutions is called a fundamental matrix for (1.2) on \( I \). Denote the fundamental matrix formed from the solutions \( \phi_1, \ldots, \phi_n \) by \( \Phi \). The statement that every solution \( \phi \) of (1.2) is the
linear combination of $\phi_1, \ldots, \phi_n$ for some unique choice of the constants $c_1, \ldots, c_n$ is simply that

\begin{equation}
(1.3) \quad \phi(t) = \Phi(t)c
\end{equation}

where $\Phi$ is the fundamental matrix solution and $c$ is the column vector with the components $c_1, \ldots, c_n$.

**Theorem 1.3.** If $\Phi$ is a solution matrix of (1.2) on $I$ and if $t_0$ is any point of $I$, then

\[ (\det \Phi)' = \left( \sum_{j=1}^{n} a_{jj}(t) \right) \det \Phi, \]

\[ \det \Phi(t) = \det \Phi(t_0) \exp \left[ \int_{t_0}^{t} \sum_{j=1}^{n} a_{jj}(s) ds \right], \quad \text{for every } t \in I. \]

It follows that either $\det \Phi(t) \neq 0$ for each $t \in I$ or $\det \Phi(t) = 0$ for every $t \in I$.

**Proof.** Because the column vectors of $\Phi$ is a solution of (1.2), we have

\begin{equation}
(1.4) \quad \phi'_{ij} = \sum_{k=1}^{n} a_{ik} \phi_{kj}, \quad i, j = 1, \ldots, n.
\end{equation}

Therefore,

\[
(\det \Phi)' = \begin{vmatrix}
\phi'_{11} & \phi'_{12} & \cdots & \phi'_{1n} \\
\phi'_{21} & \phi'_{22} & \cdots & \phi'_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi'_{n1} & \phi'_{n2} & \cdots & \phi'_{nn}
\end{vmatrix}
+ \begin{vmatrix}
\phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\
\phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n1} & \phi_{n2} & \cdots & \phi_{nn}
\end{vmatrix} + \cdots
\]
Using (1.4), we have

\[
(det \Phi)' = \left| \begin{array}{cccc}
\sum_{k=1}^n a_{1k} \phi_{k1} & \sum_{k=1}^n a_{1k} \phi_{k2} & \cdots & \sum_{k=1}^n a_{1k} \phi_{kn} \\
\phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \\
\end{array} \right| + \cdots \\
+ \left| \begin{array}{cccc}
\sum_{k=1}^n a_{2k} \phi_{k1} & \sum_{k=1}^n a_{2k} \phi_{k2} & \cdots & \sum_{k=1}^n a_{2k} \phi_{kn} \\
\phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \\
\sum_{k=1}^n a_{nk} \phi_{k1} & \sum_{k=1}^n a_{nk} \phi_{k2} & \cdots & \sum_{k=1}^n a_{nk} \phi_{kn} \\
\end{array} \right| + \cdots 
\]

Using elementary row elimination, we find that the first row of the first determinant simplifies to

\[a_{11} \phi_{11} a_{11} \phi_{12} \cdots a_{11} \phi_{1n}.\]

Similarly for the ith row of the ith determinant. Thus

\[(det \Phi)' = a_{11} det \Phi + a_{22} det \Phi + \cdots + a_{nn} det \Phi\]

for every \(t \in I\). This proves the first part of the theorem.

The rest of the proof follows by solving the scalar equation for \(det \Phi\). \(\square\)

**Theorem 1.4.** A solution matrix \(\Phi\) of (1.2) on an interval \(I\) is a fundamental matrix of (1.2) if and only if \(det \Phi(t) \neq 0\) for every \(t \in I\).

**Theorem 1.5.** If \(\Phi\) is a fundamental matrix for (1.2) on \(I\) and \(C\) is a nonsingular constant matrix, then \(\Phi C\) is also a fundamental matrix for (1.2). Every fundamental matrix of (1.2) is of this form for some nonsingular matrix \(C\).

### 1.2. Linear Nonhomogeneous Systems

Consider the linear nonhomogeneous system as in (1.1)

\[(1.1) \quad y' = A(t)y + g(t).\]

Suppose \(\phi_1\) and \(\phi_2\) are any two solutions of (1.1) on \(I\). Then \(\phi_1 - \phi_2\) is a solution of the associated homogeneous systems (1.2) on \(I\). By the remark following Theorem 1.2, there exists a constant vector \(c\) such that

\[\phi_1 - \phi_2 = \Phi c.\]

The general solutions for (1.1) are

\[(1.5) \quad \psi = \Phi c + \phi_0,\]

where \(\Phi\) is a fundamental matrix solution of (1.2), \(c\) is an arbitrary constant vector and \(\phi_0\) is a particular solution of (1.1). If the initial condition \(\psi(t_0) = \eta\) is given for \(t_0 \in I\), then, the constant vector \(c\) can be solved from the given vector \(\eta\). Note that the matrix \(\Phi(t_0)\) is nonsingular.
Theorem 1.6. If $\Phi$ is a fundamental matrix of (1.2) on $I$, then the function

$$
\psi(t) = \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) g(s) ds
$$

is the (unique) solution of (1.1) valid on $I$ and satisfying the initial condition

$$
\psi(t_0) = 0.
$$

If the initial condition $\psi(t_0) = \eta$ is given, then

$$
\phi(t) = \phi_h(t) + \psi(t),
$$

where $\psi$ is the solution given in Theorem 1.6, and $\phi_h$ is the solution of the homogeneous system (1.2) satisfying the initial condition $\phi_h(t_0) = \eta$, the same $\eta$ as the initial condition for $\phi$.

We have

$$
\phi(t) = \Phi(t) \Phi^{-1}(t_0) \eta + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s) g(s) ds.
$$

Let $T(t, s) = \Phi(t) \Phi^{-1}(s)$. Then $T(t, s)$ is a special fundamental matrix solution that satisfies $T(s, s) = I$, and is called the principal matrix solution. We can write the solution to (1.1) as

$$
\phi(t) = T(t, t_0) \eta + \int_{t_0}^{t} T(t, s) g(s) ds.
$$

2. Linear Systems with Periodic Coefficients

2.1. Homogeneous systems with Periodic Coefficients. Consider the homogeneous system

$$
(1.2) \quad y' = A(t)y,
$$

where $A(t)$ is a continuous periodic $n \times n$ matrix of period $\omega$, i.e., $A(t + \omega) = A(t), t \in \mathbb{R}$. If $\Phi(t)$ is a fundamental matrix for (1.2), then $\Phi(t + \omega)$ is also a fundamental matrix of (1.2). Therefore there exists a nonsingular constant matrix $C$ such that

$$
\Phi(t + \omega) = \Phi(t) C, \quad -\infty < t < \infty.
$$

In the rest of this paper, we assume that $\Phi(t)$ satisfies the initial condition $\Phi(0) = I$. Then we have

$$
\Phi(t + \omega) = \Phi(t) \Phi(\omega), \text{ proof?}
$$

The nonsingular matrix $C = \Phi(\omega)$ is called the monodromy matrix. Since $C$ is nonsingular, there exists a matrix $R$ such that $C = \exp(\omega R)$. Note that

$$
(2.1) \quad \Phi(\omega) = \exp(\omega R).
$$

Theorem 2.1. Let $A(t)$ be a continuous periodic matrix of period $\omega$ and let $\Phi(t)$ be any fundamental matrix of (1.2). Then there exists a periodic nonsingular matrix $P(t)$ of period $\omega$ and a constant matrix $R$ such that

$$
\Phi(t) = P(t) \exp(tR).
$$

Remark 2.1. (1) Note that $\Phi(t)$ may not be periodic.

(2) If $A(t)$ is a constant matrix, hence periodic of any period $\omega$, then $P(t) = I$ and $R = A$. 

Proof. Let $R$ be as in (2.1). Define

$$P(t) = \Phi(t) \exp(-tR), \quad -\infty < t < \infty.$$ 

$P(t)$ is nonsingular and

$$P(t + \omega) = \Phi(t + \omega) \exp(-(t + \omega)R)$$

$$= \Phi(t) C \exp(-tR) \exp(-\omega R)$$

$$= \Phi(t) \exp(-tR) = P(t).$$

□

The Floquet theorem can be used to transform (1.2) to a linear system with constant coefficients. Let

$$y = P(t)u. \quad (2.2)$$

Since $y$ is a solution and $\Phi(t)$ is a fundamental matrix to (1.2), there exits a constant $c$ such that

$$y = \Phi(t)c$$

$$= P(t) \exp(tR)c$$

$$= P(t)u,$$

it follows that

$$u = \exp(tR)c.$$

Therefore

$$u' = Ru. \quad (2.3)$$

Corollary 2.2. The change of variable $y = P(t)u$ transforms the periodic system (1.2) to the system (2.3) with constant coefficients.

Corollary 2.3. A nontrivial solution $\phi(t)$ of (1.2) has the property

$$\phi(t + \omega) = k \phi(t), \quad -\infty < t < \infty,$$

where $k$ is a constant, if and only if $k$ is an eigenvalue of $\Phi(\omega) = \exp(\omega R)$.

The eigenvalues $\rho_i$ of the nonsingular matrix $\Phi(\omega) = \exp(\omega R)$ is called the characteristic multipliers of the system (1.2), and the eigenvalues $\lambda_i$ of the matrix $R$ is called the characteristic exponents of (1.2). One can show that

$$\rho_j = e^{\omega \lambda_j}, \quad \lambda_j = \frac{1}{\omega} \log \rho_j, \quad (\text{mod} \frac{2\pi i}{\omega}), \quad j = 1, \ldots, n.$$ 

$$\prod_{j=1}^{n} \rho_j = \exp \left( \int_{0}^{\omega} trA(s) ds \right), \quad \sum_{j=1}^{n} \lambda_j = \frac{1}{\omega} \int_{0}^{\omega} trA(s) ds, \quad \left(\text{mod} \frac{2\pi i}{\omega}\right).$$

Corollary 2.4. If the characteristic exponents of (1.2) have negative real parts (or equivalently, it the multipliers (1.2) have magnitude strictly less than 1), then all solutions of (1.2) approach zero as $t \to \infty$. 

2.2. Nonhomogeneous Systems with Periodic Coefficients. We consider the nonhomogeneous system

\[(1.1) \quad y' = A(t)y + g(t),\]

where we assume that \(A(t)\) and \(g(t)\) are continuous and periodic in \(t\) of the same period \(\omega\).

**Theorem 2.5.** A solution \(\phi(t)\) of (1.1) is periodic of period \(\omega\) if and only if \(\phi(\omega) = \phi(0)\).

**Proof.** The condition is clearly necessary. To prove the sufficiency, let \(\psi(t) = \phi(t + \omega)\). Then \(\phi\) and \(\psi\) are both solutions of (1.1) and \(\psi(0) = \phi(\omega) = \phi(0)\). By the uniqueness theorem, \(\phi(t) = \psi(t) = \phi(t + \omega)\) for all \(t\). \(\square\)

**Theorem 2.6.** The system (1.1) has a periodic solution of period \(\omega\) for any periodic forcing vector \(g\) of period \(\omega\) if and only if the homogeneous system (1.2) has no periodic solution of period \(\omega\) except for the trivial solution \(y = 0\).

**Proof.** Since \(\Phi(t)\) be a fundamental matrix of (1.2) with \(\Phi(0) = I\). A solution of (1.1) has the form

\[\psi(t) = \Phi(t)\psi(0) + \Phi(t) \int_0^t \Phi^{-1}(s)g(s)ds.\]

The solution \(\psi\) is periodic of and only if \(\psi(0) = \psi(\omega)\). But

\[\psi(\omega) = \Phi(\omega)\psi(0) + \Phi(\omega) \int_0^\omega \Phi^{-1}(s)g(s)ds.\]

Therefore, \(\psi(0) = \psi(\omega)\) if and only if

\[\[I - \Phi(\omega)]\psi(0) = \Phi(\omega) \int_0^\omega \Phi^{-1}(s)g(s)ds.\]

The homogeneous algebraic system must be solved for every periodic forcing term \(g\). This is possible if and only if \(det(I - \Phi(\omega)) \neq 0\). On the other hand, (1.2) has a periodic solution of period \(\omega\) if and only if 1 is a multiplier of \(y' = A(t)y\). Thus, \(det(I - \Phi(\omega)) \neq 0\) if and only if (1.2) has no nontrivial periodic solution of period \(\omega\). \(\square\)

**Theorem 2.7.** (Fredholm’s alternative). If \(A\) and \(g\) are continuous and periodic in \(t\) with period \(\omega\), then (1.1) has a period \(\omega\) solution if and only if

\[\int_0^\omega z(t)g(t)dt = 0,\]

for all period \(\omega\) solutions \(z\) of the adjoint equation

\[\dot{z} = -zA(t).\]

Note that \(z(t)\) is a row vector in the adjoint equation.