A Geometric Proof for the Jordan Canonical Form of a matrix $A$

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Let $A$ be an $n \times n$ matrix. We say that $u$ is an eigenvector corresponding to the eigenvalue $\lambda$ if

$$(A - \lambda I)u = 0.$$ 

We say $u$ is a generalized eigenvector if there exists $N > 1$ such that

$$(A - \lambda I)^N u = 0.$$ 

The smallest of such $N$ is called the flag of the generalized eigenvector $u$. If $u$ is a generalized eigenvector of flag $N$, then the sequence of vectors

$$u, (A - \lambda I)u, \ldots, (A - \lambda I)^{N-1}u,$$

are all generalized eigenvectors of descending flags. The last entry $(A - \lambda I)^{N-1}u$ is of flag one and is just a regular eigenvector. One can verify that the vectors in the sequence are all linearly independent.

**Theorem 0.1.** [Jordan Canonical form] There exists a basis of $n$ generalized eigenvectors

$$P = (u_1, \ldots, u_n)$$

such that

$$P^{-1}AP = \text{diag}(B_i), \quad i = 1, \ldots, m$$

where $B_i$ is a square matrix of the form

$$B_i = \begin{pmatrix}
\lambda_i & 1 & 0 & \cdots & 0 \\
0 & \lambda_i & 1 & \cdots & 0 \\
0 & 0 & \lambda_i & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_i
\end{pmatrix} = \text{diag}(\lambda_i) + N_i.$$ 

The matrices $N_i, i = 1, \ldots, m$, is nilpotent.

See http://en.wikipedia.org/wiki/Nilpotent_matrix

The proof of Theorem 0.1 is divided into several lemmas, which are of independent interest.

For each $\lambda \in \mathbb{C}$, let $A_\lambda = A - \lambda I$. Let $X = \mathbb{C}^n$.

**Lemma 0.2.**

$$X \supset R(A_\lambda) \supset \cdots \supset R(A_\lambda^k) \supset R(A_\lambda^{k+1}),$$

$$R(A_\lambda^{k+1}) = A_\lambda R(A_\lambda^k)$$
Lemma 0.3.

\[ 0 \in N(A_\lambda) \subset N(A_\lambda^2) \subset \cdots \subset N(A_\lambda^k) \subset N(A_\lambda^{k+1}), \]

\[ N(A_\lambda^{k+1}) = A_\lambda^{-1} N(A_\lambda^k), \]

where \( A_\lambda^{-1} \) denotes the pre-image of the mapping \( A_\lambda \).

Definition 0.4. There exists \( k \geq 0 \) such that \( N(A_\lambda^k) = N(A_\lambda^{k+1}) \). The smallest such number is called the ascent of \( A_\lambda \), and is denoted by \( \alpha(A_\lambda) \).

There exists \( k \geq 0 \) such that \( R(A_\lambda^k) = R(A_\lambda^{k+1}) \). The smallest such number is called the descent of \( A_\lambda \), and is denoted by \( \delta(A_\lambda) \).

Remark 0.5. (1) If \( \alpha(A_\lambda) = 0 \), then \( A_\lambda \) is nonsingular. If \( \delta(A_\lambda) = 0 \), then \( R(A_\lambda) = X \). Again \( A_\lambda \) is nonsingular. Thus, \( \alpha(A_\lambda) = 0 \) is equivalent to \( \delta(A_\lambda) = 0 \).

(2) \( R(A_\lambda^k) = R(A_\lambda^{k+1}) \) for all \( k \geq \delta(A_\lambda) \). Similarly \( N(A_\lambda^k) = N(A_\lambda^{k+1}) \) for all \( k \geq \alpha(A_\lambda) \).

Lemma 0.6. \( \alpha := \alpha(A_\lambda) = \delta := \delta(A_\lambda) \).

Proof. The proof is divided into two parts:

(1) Show \( \alpha \leq \delta \).

Since \( N(A_\lambda^{\alpha-1}) \neq N(A_\lambda^\alpha) \), there exists \( x_0 \in N(A_\lambda^\alpha) \) but \( x_0 \neq N(A_\lambda^{\alpha-1}) \). Thus, \( y = A_\lambda^{\alpha-1} x_0 \neq 0 \) but \( A_\lambda^\alpha x_0 = 0 \). This shows that \( \dim R(A_\lambda^{\alpha-1}) < \dim R(A_\lambda^\alpha) \). Thus \( R(A_\lambda^{\alpha-1}) \neq R(A_\lambda^\alpha) \). Therefore \( \delta(A_\lambda) \geq \alpha(A_\lambda) \).

(2) Show \( \delta \leq \alpha \).

Since \( R(A_\lambda^{\delta-1}) \neq R(A_\lambda^\delta) \), there exists \( y \in R(A_\lambda^{\delta-1}) \), \( y \neq 0 \) but \( A_\lambda y = 0 \). Let \( y = A_\lambda^{\delta-1} x_0 \). Then \( A_\lambda^{\delta-1} x_0 \neq 0 \) but \( A_\lambda^\delta x_0 = 0 \). This shows that \( N(A_\lambda^{\delta-1}) \neq N(A_\lambda^\delta) \). Therefore \( \alpha(A_\lambda) \geq \delta(A_\lambda) \).

Lemma 0.7. Let \( \lambda \) be an eigenvalue of \( A \), and \( p := \alpha(A_\lambda) = \delta(A_\lambda) \). Then:

(1) The matrix \( A_\lambda \) is nonsingular in \( R(A_\lambda^p) \).

(2) Let \( \mu \neq \lambda \). Then the matrix \( A_\mu \) is nonsingular in \( N(A_\lambda^p) \).

Proof. (1) If there exists \( x \in R(A_\lambda^p) \) such that \( A_\lambda x = 0 \), then \( A_\lambda R(A_\lambda^p) \neq R(A_\lambda^p) \), contradicting to \( p = \delta(A_\lambda) \).

(2) Assume that there exists \( x \in N(A_\lambda^p) \) such that \( A_\mu x = 0 \). Since \( x \in N(A_\lambda^p) \), there exists an integer \( k \geq 0 \) such that \( A_\lambda^{k-1} x \neq 0 \), \( A_\lambda^k x = 0 \). Applying \( A_\lambda^{k-1} \) to

\[ A_\lambda x + (\lambda - \mu) x = 0, \]

we have \( (\lambda - \mu) A_\lambda^{k-1} x = 0 \). This can occur only if \( \lambda - \mu = 0 \).
Our next main result shows that $X$ is a direct sum of two invariant subspaces $R(A^p_\lambda) \oplus N(A^p_\lambda) = X$, where $\lambda$ is nonsingular on $R(A^p_\lambda)$ and $\lambda$ is the only eigenvalue for $N(A^p_\lambda)$.

**Lemma 0.8.** Let $p = \alpha(A_\lambda) = \delta(A_\lambda)$. Then
\[ R(A^p_\lambda) \oplus N(A^p_\lambda) = X. \]

**Proof.** (1) We show that $R(A^p_\lambda) \cap N(A^p_\lambda) = 0$.
If $y \in R(A^p_\lambda) \cap N(A^p_\lambda)$, then $y = A^p_\lambda x$ and $A^p_\lambda y = 0$. Thus $A^{2p}_\lambda x = 0$. $x \in N(A^{2p}_\lambda) = N(A^p_\lambda)$. Thus $A^p_\lambda x = 0$. This implies $y = 0$.

(2) We show that for any $x \in X$, there exist $x_1 \in R(A^p_\lambda)$, $x_2 \in N(A^p_\lambda)$ such that $x = x_1 + x_2$.
Consider $A^p_\lambda x \in R(A^p_\lambda) = R(A^{2p}_\lambda)$. There exists $y$ such that $A^{2p}_\lambda y = A^p_\lambda x$. Thus
\[ A^p_\lambda (x - A^p_\lambda y) = 0. \]
Let $x_1 = A^p_\lambda y$, and $x_2 = x - x_1$. Then $x_1 \in R(A^p_\lambda)$, $x_2 \in N(A^p_\lambda)$. □

Let
\[ \lambda_1, \ldots, \lambda_m \]
be a list of all the distinct eigenvalues of $A$, each associated with a unique
\[ p_i = \alpha(A_{\lambda_i}) = \delta(A_{\lambda_i}). \]
We have shown
\[ R(A^{p_i}_{\lambda_i}) \oplus N(A^{p_i}_{\lambda_i}) = X. \]
Clearly $N(A^{p_i}_{\lambda_i})$ is the generalized eigenspace for $\lambda_i$.

We now show

**Lemma 0.9.**
\[ \oplus_i N(A^{p_i}_{\lambda_i}) = X. \]

**Proof.** (1) We show that the generalized eigenspaces, each corresponding to a distinct $\lambda_i$, are linearly independent. Let $k$ be the smallest integer such that $x_1 + \cdots + x_k = 0$ where we assume, without loss of generality, $x_j \in N(A^{p_j}_{\lambda_j})$ and is nonzero.

Applying $A^{p_i}_{\lambda_i}$ to $x_1 + \cdots + x_k = 0$, we have
\[ A^{p_i}_{\lambda_i}x_2 + \cdots + A^{p_i}_{\lambda_i}x_k = 0. \]
From Lemma 0.7, $A^{p_i}_{\lambda_i}$ is nonsingular in each of its invariant subspace $N(A^{p_i}_{\lambda_i})$, $j \neq 1$, and thus $y_j := A^{p_i}_{\lambda_i}x_j \neq 0$. We have $y_2 + \cdots + y_k = 0$. This is a contradiction to the minimal property of $k$.

(2) For any $x \in X$, we show, by induction, that it is possible to express $x$ as $x = x_1 + \cdots x_m$ with $x_i \in N(A^{p_i}_{\lambda_i})$. 
If $m = 1$, then $R(A_{\lambda_1}^p) \oplus N(A_{\lambda_1}^p)$. From Lemma 0.7, $\lambda_1$ is not an eigenvalue of $A$ restricted to $R(A_{\lambda_1}^p)$. If $R(A_{\lambda_1}^p) \neq 0$, then the restriction of $A$ to it has no eigenvalue. This is a contradiction. Thus $R(A_{\lambda_1}^p) = 0$ and $X = N(A_{\lambda_1}^p)$.

If the lemma is true for $m - 1$, then consider all the invariant subspaces $N(A_{\lambda_2}^p), \ldots, N(A_{\lambda_m}^p)$.

They are all contained in $R(A_{\lambda_1}^p)$ because from Lemma 0.7, $A_{\lambda_1}$ is nonsingular in each $N(A_{\lambda_j}^p)$, $j \geq 2$. Therefore, if $A$ is restricted to $R(A_{\lambda_1}^p)$, it has all the eigenvalues $\lambda_2, \ldots, \lambda_m$ but not $\lambda_1$. This is due to Lemma 0.7 that $\lambda_1$ is not an eigenvalue in $R(A_{\lambda_1}^p)$. Based on the induction assumption, for the case of $m - 1$ eigenvalues, we have

$$R(A_{\lambda_1}^p) = \bigoplus_{2 \leq j \leq m} N(A_{\lambda_j}^p).$$

Thus

$$X = \bigoplus_{1 \leq j \leq m} N(A_{\lambda_j}^p).$$

\[\square\]

**Figure 1.** A cyclic basis can be selected for $N(A_{\lambda}^p)$. To finish the proof of Theorem 0.1, it remains to show that in each generalized eigenspace $N(A_{\lambda_j}^p)$, a cyclic basis can be selected. The idea of the proof is illustrated in Figure 1. The top, left box holds generalized eigenvectors of flag $p$, pick $u = x_i^p$, $i = 1, \ldots, q_1$ from that box, then $(A_{\lambda_1} u, A_{\lambda_1}^2 u, \ldots, A_{\lambda_1}^{p-1} u)$ are placed sequentially below on the first column of boxes. Similarly the top, second-from-left box contains generalized eigenvectors of flag $p-1$. Pick $u = x_i^p$, $i = q_1 + 1, \ldots, q_2$ from that box and apply powers of $A_{\lambda_2}$ to $u$ generates the second column of boxes, etc. Note the second column has only $p - 1$ boxes because the flag number of the vectors from the top box is smaller.
However, it is impossible to find vectors in the top, left box since vectors of flag \( p \) do not form a linear subspace. Instead, we start by finding vectors in the bottom box of the first column since they do form the linear subspace \( R(A_{n-1}^p) \). This suggests the following construction.

Let \( T \) be the restriction of the matrix \( A_\lambda \) to \( Y := N(A_{n}^p) \), where \( \lambda \) is one of the \( \lambda_j \) and \( p = p_j \). Since \( T^p = 0 \) but \( T^{p-1} \neq 0 \), we have \( R(T^{p-1}) \neq \{0\} \). Let \( (x_1^1, \ldots, x_{q_n}^1) \) be a basis of \( R(T^{p-1}) \). Each \( x_i^1 \) can be written as \( x_i^1 = T^{p-1} x_i^0 \) for some \( x_i^0 \in Y, i = 1, \ldots, q_1 \). If \( p > 1 \), set \( T^{p-2} x_i^p = x_i^2 \), so that \( T x_i^2 = x_i^1 \). The vectors \( x_i^k, k = 1, 2, i = 1, \ldots, q_1 \), belong to \( R(T^{p-2}) \) and are linearly independent. In fact, \( \sum \alpha_i x_i^2 + \sum \beta_i x_i^1 = 0 \) implies \( \sum \alpha_i x_i^2 = 0 \) on applying \( T \) and hence \( \alpha_i = 0 \) for all \( i \), hence \( \sum \beta_i x_i^1 = 0 \) and \( \beta_i = 0 \) for all \( i \). We can enlarge the family \( \{x_i^k\}, k = 1, 2, \) to a basis of \( R(T^{p-2}) \) by adding, if necessary, new vectors \( x_{q_1+1}^2, \ldots, x_{q_2}^2 \); here we can arrange that \( T x_i^2 = 0 \) for \( i > q_1 \). (Proof: Consider \( T x_i^2 \), \( i > q_1 \), which are in \( R(T^{p-1}) \). Therefore \( T x_i^2 \), \( i > q_1 \) can be written as a linear combination of \( x_i^1, i = 1, \ldots, q_1 \). We then subtracting \( x_i^2 \), \( i > q_1 \), by a linear combination of \( x_i^1, i = 1, \ldots, q_1 \) with the same coefficients. This yields a revised set of new vectors which are mapped to 0 by \( T \).

If \( p > 2 \), we can repeat the process the same way. Finally, we arrive at a basis \( \{x_i^k\} \) of \( Y \) with the following properties: For each \( k = 1, \ldots, p \), the index \( j = 1, \ldots, q_k \), where \( q_1 \leq q_2 \leq \cdots \leq q_p \). Moreover,

\[
T x_j^k \begin{cases} x_j^{k-1}, & 1 \leq j \leq q_k-1, \\ 0, & q_k-1 + 1 \leq j \leq q_k, \end{cases}
\]

where we set \( q_0 = 0 \).

If we arrange the basis \( \{x_j^k\} \) in the order \( \{x_1^1, x_2^1, x_1^p, x_2^p, \ldots\} \), the matrix of \( T \) with respect to this basis takes the form

\[
\begin{pmatrix}
0 & 1 & \cdots & 0 & 1 \\
0 & 1 & \cdots & 0 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 1 \\
0 & 1 & \cdots & 0 & 1
\end{pmatrix}
\]