1. (20 points) Find the orthogonal complement of the subspace of \( \mathbb{R}^3 \) spanned by the vectors \((1, 2, 1)^T\) and \((1, -1, 1)^T\).

**Solution 1.** The two vectors are linearly independent, it suffices therefore to find a vector orthogonal to both. The first and last coordinates of these two vectors coincide so one might want to test if the vector \((1, 0, -1)^T\) is the desired vector. Call it... a hunch.

If you did not notice this, the fastest way to compute the orthogonal complement is to find the null space of the matrix

\[
A = \begin{bmatrix}
1 & 2 & 1 \\
1 & -1 & 1
\end{bmatrix}
\]

i.e. to solve the system \(Ax = 0\). It is indeed the subspace of \( \mathbb{R}^3 \) spanned by \((1, 0, -1)^T\).

2. (20 points) Write down the normal equations and find all the least square solutions to the linear system \(Ax = b\) where

\[
A = \begin{bmatrix}
1 & 2 \\
2 & 4 \\
-1 & -2
\end{bmatrix}
\]

and

\[
b = \begin{bmatrix}
3 \\
2 \\
1
\end{bmatrix}
\]

**Solution 2.** The normal equations are \(A^TAx = A^Tb\), i.e.

\[
\begin{bmatrix}
6 & 12 \\
12 & 24
\end{bmatrix}x = \begin{bmatrix}
6 \\
12
\end{bmatrix}
\]

Reducing to row-echelon form we get

\[
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}x = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

which is satisfied by any vector of the form \(x = (1 - 2\alpha, \alpha)^T\).

3. (20 points) Let \(x = (-3, 4, 0)^T\). Compute \(||x||_1||, ||x||_2||, and ||x||_\infty||.

**Solution 3.**

- \(||x||_1|| = |-3| + |4| + |0| = 7
- \(||x||_2|| = \sqrt{(-3)^2 + 4^2 + 0^2} = \sqrt{25} = 5
- \(||x||_\infty|| = \max{|-3|, |4|, |0|} = 4.

4. (20 points) Use the Gram-Schmidt process to find an orthonormal basis for the subspace of \( \mathbb{R}^4 \) spanned by the vectors \(x_1 = (1, 1, -1, 1)^T\), \(x_2 = (2, 0, 0, 2)^T\), and \(x_3 = (4, 2, 2, 0)^T\).
Solution 4. Since $\|x_1\| = 2$ we have $u_1 = \frac{1}{2}x_1$. The projection of $x_2$ onto the space spanned by $u_1$ is

$$p_1 = \langle x_2, u_1 \rangle u_1 = 2u_1 = x_1$$

Therefore $x_2 - p_1 = x_2 - x_1 = (1, -1, 1)^T$. Its length is $\|x_2 - p_1\| = 2$ and so $u_2 = \frac{1}{2}(1, -1, 1)^T$. The projection of $x_3$ on the space spanned by $u_1$ and $u_2$ is

$$p_2 = \langle x_3, u_1 \rangle u_1 + \langle x_3, u_2 \rangle u_2 = 2u_1 + 2u_2 = (2, 0, 2)^T$$

so $x_3 - p_2 = (2, 2, -2)^T$. We have $\|x_3 - p_2\| = 4$ and $u_3 = \frac{1}{2}(1, 1, -1)^T$.

5. (20 points) Recall that two $n \times n$ matrices $A$ and $B$ are said to be similar if there exists an invertible matrix $S$ such that $A = S^{-1}BS$. Recall also that the trace of a matrix $A$ is

$$\text{Tr}(A) := \sum_{i=1}^{n} a_{ii}.$$ 

- Show that for any two matrices $A$ and $B$

$$\text{Tr}(AB) = \text{Tr}(BA).$$

- Use the above result to show that if $A$ and $B$ are similar then

$$\text{Tr}(A) = \text{Tr}(B)$$

(you can assume that the result holds even if you did not manage to prove it).

Solution 5. 

- Recall that the $ij$-entry of $AB$ is

$$\sum_{k=1}^{n} a_{ik} b_{kj}.$$ 

We get

$$\text{Tr}(AB) = \sum_{i=1}^{n} \left( \sum_{k=1}^{n} a_{ik} b_{ki} \right) = \sum_{k=1}^{n} \left( \sum_{i=1}^{n} b_{ki} a_{ik} \right) = \text{Tr}(BA).$$

- Using the above result we get

$$\text{Tr}(A) = \text{Tr}(S^{-1}BS) = \text{Tr}(S^{-1}(BS)) = \text{Tr}((BS)S^{-1}) = \text{Tr}(B)$$

6. (5 bonus points; do not work on this unless you are done with the rest and you are confident that what you did is correct.)

Let $U$ be an $m$-dimensional subspace of $\mathbb{R}^n$ and let $V$ be a $k$-dimensional subspace of $U$, where $0 < k < m$. Show that any orthonormal basis

$$\{v_1, \ldots, v_k\}$$

of $V$ can be expanded to form an orthonormal basis

$$\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_m\}$$

of $U$.

Solution 6. We saw in class that any collection of linearly independent vectors

$$\{v_1, \ldots, v_k\}$$

in a vector space $U$ can be completed to a basis of $U$. Let

$$\{v_1, \ldots, v_k, w_{k+1}, \ldots, w_m\}$$

be such a completion. Applying the orthonormalization process to it we do not modify the first $k$ vectors since they are already orthonormal and we obtain an orthonormal basis of $U$. 