1. Crystal bases

Example 1.1. $\mathfrak{g} = \mathfrak{sl}_3$. We represent the adjoint representation as the following graph.

```
  O --- F2 --- O
  |     |     |
  F1   F1   F1
  |     |     |
  O --- F2 --- O
```

Each vertex corresponds to a weight space, but $\bullet$ has 2 dimensions, and there is a priori no way to find a distinguished basis.

To fix this:

1. Choose a clever basis
2. Study representations of $U_q(\mathfrak{sl}_3)$
3. Modify Chevalley generators $F_1, F_2$ (and $E_1, E_2$) to get operations $\tilde{F}_1, \tilde{F}_2$ (and $\tilde{E}_1, \tilde{E}_2$)
4. Take the limit as $q \to 0$.

To modify the Chevalley generators, we can look to the case $\mathfrak{g} = \mathfrak{sl}_2$. In this case, $F$ does not act as a partial permutation of basis elements because we have to introduce coefficients like 2. There is a simple fix: just replace $F$ by $\tilde{F}$ which is the obvious partial permutation that ignores these coefficients. A naive way to do it in general is to restrict to each color and make this modification. Miraculously, this idea works.

The resulting structure is the crystal graph of the representation.

Theorem 1.2 (Kashiwara). Let $\mathfrak{g}$ be any symmetrizable Kac–Moody algebra and $V$ any highest weight representation. Then there is always a basis of $V$ where the above works.
When $\mathfrak{g} = \mathfrak{sl}_2$, crystals are just directed intervals $B(n)$ with $n + 1$ nodes, for each $n \in \mathbb{Z}_{\geq 0}$. The tensor product rule for $B(3) \otimes B(2)$ is expressed graphically as follows:

```
  o → o → o → o
  ↓    ↓    ↓    ↓
  o → o → o → o
  ↓    ↓    ↓    ↓
  o → o → o → o
```

The first tensor factor is placed at the top, and the second on the left side. The vertices of the tensor product are all pairs of vertices one in each factor, which can be arranged in a grid as shown. The top row and right column of the grid make one irreducible component. Then the nodes second from the top and second from the right. Continue until all the nodes are used up.

For other $\mathfrak{g}$, simply treat the arrows coming from each copy of $U_q(\mathfrak{sl}_2)$ separately. Decompose the crystal into connected components for the arrows corresponding to that $\tilde{F}_i$, and take the tensor product of each pair. Once this has been done for all the different $\tilde{F}_i$ (i.e., all different colors of arrows in the crystal graph), the result is the tensor product of the $\mathfrak{g}$ crystals.

**Example 1.3.** Consider $\mathfrak{g} = \mathfrak{sl}_3$. Then the two fundamental crystals are $B(\omega_1) = \circ \quad 1 \rightarrow \circ \quad 2 \rightarrow \circ$ and $B(\omega_2) = \circ \quad 2 \rightarrow \circ \quad 1 \rightarrow \circ$. Their tensor product is

```
  1 → 2 → o
  ↓   ↓
  2 → 1 → o
  ↓   ↓
  1 → 1 → o
  ↓   ↓
  2 → o
```

which illustrates that $V(\omega_1) \otimes V(\omega_2) \cong V(\omega_1 + \omega_2) \oplus V(0)$.

# 2. Global / canonical bases

Recall the Verma module construction: $V(\lambda) = U_q^{-}(\mathfrak{g})/I(\lambda)$. We can define $\tilde{F}_i$ on $U_q^{-}(\mathfrak{g})$ by $\lim_{\lambda \to \infty} \tilde{F}_i$ on $V(\lambda)$.

Get a notion of crystal basis / crystal for $U_q^{-}(\mathfrak{g})$. It is a fact that these crystal bases for $U_q^{-}(\mathfrak{g})$ are essentially unique.

**2.1. More amazing facts.** There is a completely unique basis $G(\infty)$ of $U_q^{-}(\mathfrak{g})$ such that

1. It contains $1$,
2. It specializes to a basis of $U_q^{-\text{res}}(\mathfrak{g})$ (divided power $\mathbb{Z}$-form) for all $q \neq 0, \infty$, i.e., matrix coefficients don’t blow up at any $q \neq 0$.
3. $G(\infty)$ is a crystal basis at $q = 0$ and $q = \infty$.

Facts:

1. The nonzero elements of $G(\infty)$ in $U_q^{-}(\mathfrak{g})/I(\lambda)$ are a basis for $V(\lambda)$ for all $\lambda$.
2. In simply-laced type, for all $b \in G(\infty)$ and all $i$, $F_ib$ is a positive sum of elements of $G(\infty)$, where positive refers to polynomials in $q$ whose coefficients are nonnegative integer coefficients.
3. Dual basis for $\mathbf{C}[N]$

Reference: See the Monday 6/13 lecture notes given by David Speyer: [http://pages.uoregon.edu/dmoseley/talks/talks.html](http://pages.uoregon.edu/dmoseley/talks/talks.html)

Assume that $\mathfrak{g}$ is of finite type.

Recall that $U^- (\mathfrak{g})$ and $\mathbf{C}[N]$ are dual vector spaces, where $N$ is the unipotent subgroup of a Lie group $G$ corresponding to $\mathfrak{g}$. To see the duality, we think of $U^- (\mathfrak{g})$ as differential operators, which act on functions in $\mathbf{C}[N]$ by taking derivatives and evaluating at 0. Let $G(\infty)^*$ be the basis dual to $G(\infty)$ under this duality.

**Example 3.1.** For $\mathfrak{g} = \mathfrak{sl}_3$, we have $N = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$, and $\mathbf{C}[N] = \mathbf{C}[x_{12}, x_{13}, x_{23}]$. In this case,

$$G(\infty) = \{ F_1^{(a)} F_2^{(b)} F_3^{(c)} , F_2^{(a)} F_1^{(b)} F_2^{(c)} \mid b \geq a + c \}$$

$$G(\infty)^* = \{ x_{12}^i x_{13}^j \Delta^k , x_{23}^i x_{13}^j \Delta^k \mid i, j, k \geq 0 \},$$

where $\Delta = x_{12} x_{23} - x_{13}$.

Any $f \in G(\infty)^*$ is a nonnegative valued function on $N^+ = \{ M \in N \mid \text{all minors of } M \text{ are positive} \}$. A related fact is that any $f \in \mathbf{C}[N]$ is a rational function in $\{ x_{12}, x_{13}, \Delta \}$ or $\{ x_{23}, x_{13}, \Delta \}$.

In particular, $x_{23}$ is a rational function in $\{ x_{12}, x_{13}, \Delta \}$, in fact

$$x_{23} = \frac{\Delta + x_{13}}{x_{12}}.$$ 

Note that these coefficients are positive. Also, there is a cluster algebra structure on $\mathbf{C}[N]$. The machinery of cluster algebras should help us understand this kind of phenomena. □