Toric Fiber Products

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Families of Ideals Parametrized by Graphs

- Let $G$ be a finite graph
- Let $R_G$ a polynomial ring associated to $G$
- Let $I_G \subseteq R_G$ an ideal associated to $G$

Problem
Classify the graphs $G$ such that $I_G$ satisfies some “nice” property.

- Often $I_G := \ker \phi_G$ for some ring homomorphism $\phi_G$.

Problem
Determine generators for $I_G$. How do they depend on the graph $G$?
Question

Are decompositions of the graphs $G = G_1 \# G_2$ reflected in the ideals $I_G = I_{G_1} \# I_{G_2}$?

How to study ideals of large graphs by breaking into simple pieces?
Let $\mathcal{A} = \{a^1, \ldots, a^r\} \subset \mathbb{Z}^d$.

Let $K[x] := K[x^i_j : i \in [r], j \in [s]]$, with $\text{deg } x^i_j = a^i$.

Let $K[y] := K[y^i_k : i \in [r], k \in [t]]$, with $\text{deg } y^i_k = a^i$.

Let $K[z] := K[z^i_{jk} : i \in [r], j \in [s], k \in [t]]$, with $\text{deg } z^i_{jk} = a^i$.

Let $\phi : K[z] \to K[x] \otimes_K K[y] = K[x, y]$ defined by

$$z^i_{jk} \mapsto x^i_j y^i_k \quad \text{for all } i, j, k.$$

**Definition**

Let $I \subseteq K[x], J \subseteq K[y]$ ideals homogeneous w.r.t. grading by $\mathcal{A}$. The **toric fiber product** of $I$ and $J$ is the ideal

$$I \times_{\mathcal{A}} J = \phi^{-1}(I + J).$$
Two Important Examples

Example (Coarse Grading)

Let $\mathbb{K}[x], \mathbb{K}[y]$ have common grading

$$\deg x_i = \deg y_j = 1 \text{ for all } i, j.$$  

Then $I \subseteq \mathbb{K}[x], J \subseteq \mathbb{K}[y]$ homogeneous, are homogeneous in the standard/coarse grading.

$I \times_A J \subseteq \mathbb{K}[z]$ is the ordinary Segre product ideal.

Example (Fine Grading)

Let $\mathbb{K}[x_1, \ldots, x_r], \mathbb{K}[y_1, \ldots, y_r]$ have common grading

$$\deg x_i = \deg y_i = e_i \text{ for all } i.$$  

Then $I \subseteq \mathbb{K}[x], J \subseteq \mathbb{K}[y]$ homogeneous, are monomial ideals.

$I \times_A J = I(z) + J(z) \subseteq \mathbb{K}[z_1, \ldots, z_r]$ is the sum of monomial ideals.
A More Complex Example

Let

\[ I = \langle x_{klm_1} x_{klm_2} - x_{klm_2} x_{klm_1} : k \in [r_1], l_1, l_2 \in [r_2], m_1, m_2 \in [r_3] \rangle \]

Let

\[ J = \langle y_{lmn_1} y_{lmn_2} - y_{lmn_2} y_{lmn_1} : l_1, l_2 \in [r_2], m_1, m_2 \in [r_3], n \in [r_4] \rangle \]

Let

\[ \deg x_{klm} = \deg y_{lmn} = e_l \oplus e_m \]

Define \( \phi : \mathbb{K}[z_{klmn} : k \in [r_1], \ldots] \rightarrow \mathbb{K}[a_{kl}, b_{km}, c_{ln}, d_{mn} : k \in [r_1], \ldots] \) by

\[ z_{klmn} \mapsto a_{kl} b_{km} c_{ln} d_{mn} \]

Then \( I \times_\mathcal{A} J = \ker \phi. \)
Suppose $I, J$ are toric ideal $I = I_B, J = I_C, B \in \mathbb{Z}^{e_1}, C \in \mathbb{Z}^{e_2}$

$$B = \{b^i_j : i \in [r], j \in [s]\} \quad C = \{c^i_k : i \in [r], k \in [t]\}$$

If $I_B$ homogeneous with respect to $A$, then there is a linear map

$$\pi_1 : \mathbb{Z}^{e_1} \to \mathbb{Z}^d, \pi_1(b^i_j) = a^i.$$

If $I_C$ homogeneous with respect to $A$, then there is a linear map

$$\pi_2 : \mathbb{Z}^{e_2} \to \mathbb{Z}^d, \pi_2(c^i_k) = a^i.$$

Then $I \times_A J$ is a toric ideal, whose vector configuration is a fiber product

$$B \times_A C.$$
**Definition**

The **codimension** of a TFP is the codimension of the toric ideal \( I_A \).

\( I \times_A J \) has codimension 0 iff \( A \) is linearly independent.

**Proposition**

Suppose \( A \) linearly independent. Let

\[
m = x_{j_1}^{i_1} \cdots x_{j_n}^{i_n} \text{ and } m' = x_{j_1'}^{i_1'} \cdots x_{j_{n'}}^{i_{n'}}.
\]

If \( \deg m = \deg m' \) then \( n = n' \) and

\[
i_1 = i_1', \ldots, i_n = i_n'.
\]
Theorem (Ohsugi, Michałek, etc.)

Let \( I \subseteq \mathbb{K}[x] \), and \( J \subseteq \mathbb{K}[y] \) be toric ideals, and \( \mathcal{A} \) linearly independent. Then

\[
\mathbb{K}[z]/(I \times_\mathcal{A} J) \text{ normal} \iff \mathbb{K}[x]/I \text{ and } \mathbb{K}[y]/J \text{ normal}.
\]

If \( f \in \mathbb{K}[x] \), homogeneous w.r.t. \( \mathcal{A} \), write

\[
f = \sum c_u x_{j_1 \cdots j_n}^{i_1 \cdots i_n}.
\]

Lift to

\[
\sum c_u z_{j_1 k_1}^{i_1} \cdots z_{j_n k_n}^{i_n}.
\]

Theorem

Let \( \mathcal{A} \) linearly independent. Then \( I \times_\mathcal{A} J \) generated by

1. Lifts of generators of \( I \) and \( J \)
2. “Obvious” quadrics

\[
z_{j_1 k_1}^{i_1} z_{j_2 k_2}^{i_2} - z_{j_1 k_2}^{i_1} z_{j_2 k_1}^{i_2}
\]
\[ \mathbb{N}A = \{ \lambda_1 \mathbf{a}^1 + \ldots + \lambda_r \mathbf{a}^r : \lambda_i \in \mathbb{N} \} \]
\[ R = \mathbb{K}[x]/I \text{ is an } \mathbb{N}A \text{ graded ring. } R = \bigoplus_{\mathbf{a} \in \mathbb{N}A} R_{\mathbf{a}} \]
\[ S = \mathbb{K}[y]/J \text{ is an } \mathbb{N}A \text{ graded ring. } S = \bigoplus_{\mathbf{a} \in \mathbb{N}A} S_{\mathbf{a}} \]

**Proposition**

Suppose \( A \) linearly independent. Then

\[ \mathbb{K}[z]/(I \times_A J) = \bigoplus_{\mathbf{a} \in \mathbb{N}A} R_{\mathbf{a}} \otimes_{\mathbb{K}} S_{\mathbf{a}}. \]

If \( \mathbb{K} = \overline{\mathbb{K}} \), then

\[ \text{Spec}(\mathbb{K}[z]/(I \times_A J)) \cong (\text{Spec}(\mathbb{K}[x]/I) \times \text{Spec}(\mathbb{K}[y]/J)) / \!/ T, \]

where \( T \) acts on \( \text{Spec}(\mathbb{K}[x]/I) \times \text{Spec}(\mathbb{K}[y]/J) \) via

\[ t \cdot (x, y) = (t \cdot x, t^{-1} \cdot y). \]
Higher Codimension Toric Fiber Products

- Not a GIT quotient
- No hope for general construction of generators
- When is normality preserved?

**Proposition**

Let $\mathbb{K} = \overline{\mathbb{K}}$. Suppose that $I = \bigcap_i P_i$, and $J = \bigcap_j Q_j$ are primary decompositions. Then

$$I \times_A J = \bigcap_{i,j} (P_i \times_A Q_j)$$

is a primary decomposition of $I \times_A J$. 
Suppose \( I \in \mathbb{K}[x] \), \( J \in \mathbb{K}[y] \), are toric ideals.

Let \( \mathcal{A} = \{a^1, \ldots, a^r\} \), Let \( \mathbb{K}[w] := \mathbb{K}[w_1, \ldots, w_r] \).

Let \( \psi_{xw} : \mathbb{K}[x] \to \mathbb{K}[w], \ x^i_j \mapsto w_i \) (similarly \( \psi_{yw} \))

**Definition**

Let

\[
\tilde{I} = \langle x^u - x^v \in I : \phi(x^u - x^v) = 0 \rangle \\
\tilde{J} = \langle y^u - y^v \in J : \phi(y^u - y^v) = 0 \rangle
\]

The ideal \( \tilde{I} \times \tilde{J} \) is the associated codimension 0 TFP.

\( \tilde{I} \times \tilde{J} \subseteq I \times J \)

\( \tilde{I} \times \tilde{J} \) is usually related (via graph theory) in a nice way to \( I \times J \).
Gluing Generators

Let
\[ f = x_{j_1}^{i_1} \cdots x_{j_n}^{i_n} - x_{j_1'}^{i_1'} \cdots x_{j_n'}^{i_n'} \in I \]
and
\[ g = y_{k_1}^{i_1} \cdots y_{k_n}^{i_n} - y_{k_1'}^{i_1'} \cdots y_{k_n'}^{i_n'} \in J \]
that is, \( \phi_{xw}(f) = \phi_{yw}(g) \). Then

\[ \text{glue}(f, g) = z_{j_1 k_1}^{i_1} \cdots z_{j_n k_n}^{i_n} - z_{j_1' k_1'}^{i_1'} \cdots z_{j_n' k_n'}^{i_n'} \in I \times A J \]

**Question**

Two natural classes of generators of \( I \times A J \): when do they suffice?
Gluing and the associated codim 0 tfp always suffice to generate $I \times _A J$. But... how to find the right binomials to glue?

Projecting a fiber onto $\ker_\mathbb{Z} A$.

Projected and connected fibers need not be compatible.
Summary of General Results on Toric Fiber Products

- **Codim 0 TFPs**
  - Can Explicitly Describe Generators/ Gröbner bases from $I$ and $J$
  - Normality Preserved for Toric Ideals
  - Geometric Interpretation as GIT Quotient

- **Arbitrary Codim TFPs**
  - Primary Decompositions “Multiply”
  - Can Explicitly Describe Generators given generators of $I$ and $J$ with special properties (toric case only)
Definition

Let $A : \mathbb{Z}^n \to \mathbb{Z}^d$ be a linear transformation. A Markov Basis for $A$ is a finite subset $B \subset \ker_{\mathbb{Z}}(A)$ such that for all $u, v \in \mathbb{N}^n$ with $A(u) = A(v)$ there is a sequence $b_1, \ldots, b_L \in B$ such that

1. $u = v + \sum_{i=1}^L b_i$, and
2. $v + \sum_{i=1}^L b_i \geq 0$ for $l = 1, \ldots, L$.

Markov bases allow us to take random walks over the set of nonnegative integral points inside of polyhedra.
Example: 2-way tables

Let $A : \mathbb{Z}^{k_1 \times k_2} \rightarrow \mathbb{Z}^{k_1 + k_2}$ such that

$$A(u) = \left( \begin{array}{c} \sum_{j=1}^{m} u_{1j}, \ldots, \sum_{j=1}^{m} u_{k_1j} ; \sum_{i=1}^{k} u_{i1}, \ldots, \sum_{i=1}^{k} u_{ik_2} \end{array} \right)$$

$$= \text{vector of row and column sums of } u$$

$\ker_{\mathbb{Z}}(A) = \{ u \in \mathbb{Z}^{k_1 \times k_2} : \text{row and columns sums of } u \text{ are } 0 \}$

Markov basis consists of the $2\binom{k_1}{2}\binom{k_2}{2}$ moves like:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$
Definition

Let \( A : \mathbb{Z}^n \to \mathbb{Z}^d \). The toric ideal \( I_A \) is the ideal

\[
\langle p^u - p^v : u, v \in \mathbb{N}^n, Au = Av \rangle \subset K[p_1, \ldots, p_n],
\]

where \( p^u = p_1^{u_1} p_2^{u_2} \cdots p_n^{u_n} \).

Theorem (Diaconis-Sturmfels 1998)

The set of moves \( B \subseteq \ker_{\mathbb{Z}} A \) is a Markov basis for \( A \) if and only if the set of binomials \( \{ p^{b^+} - p^{b^-} : b \in B \} \) generates \( I_A \).

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 \\
-1 & 0 & 1 & 0
\end{pmatrix}
\quad \rightarrow 
\quad p_{21} p_{33} - p_{23} p_{31}
\]
More Complicated Marginals

- Let $\Gamma = \{F_1, \ldots, F_r\}$, each $F_i \subseteq \{1, 2, \ldots, n\}$.
- Let $d = (d_1, \ldots, d_n)$ and $u \in \mathbb{Z}_{\geq 0}^{d_1 \times \cdots \times d_n}$.
- Let $A_{\Gamma, d}(u) = (u|_{F_1}, \ldots, u|_{F_r})$, lower order marginals.

2-way margins of 4-way table:
- $\{1, 2\}$
- $\{2, 3\}$
- $\{3, 4\}$
- $\{1, 4\}$ -margins

Question

How does the Markov basis of $A_{\Gamma, d}$ depend on $\Gamma$ and $d$?
How do the generators of $I_{\Gamma, d}$ depend on $\Gamma$ and $d$?
Theorem (Dobra-Sullivant 2004, Hoşten-Sullivant 2002)

Suppose that $\Gamma = \Gamma_1 \cup \Gamma_2$, and $\Gamma_1 \cap \Gamma_2$ is a face of both. Then

$$I_{\Gamma,d} = I_{\Gamma_1,d_1} \times A I_{\Gamma_2,d_2}$$

and $A$ is linearly independent.

These complexes are called reducible in statistics.
Allows for direct construction of Markov bases of reducible models.
Decomposing Along Arbitrary Subcomplexes

Theorem

Suppose that $\Gamma = \Gamma_1 \cup \Gamma_2$, and $\Gamma_1 \cap \Gamma_2 = \Delta$. Then

$$I_{\Gamma,d} = I_{\Gamma_1,d_1} \times A_{\Delta,d'} I_{\Gamma_2,d_2},$$

and, associated codim zero TFP is $I_{\Gamma \cup 2|\Delta|,d'}$.

If overlap is “small”, and associated codim zero TFP is “simple”, can construct generators of $I_{\Gamma,d}$.
Example Applications

Theorem (Kral, Norin, Pragnac 2010)

If $d_i = 2$ for all $i$, then $I_{\Gamma,d}$ is generated in degree $\leq 4$ if $\Gamma$ is a series-parallel graph (no $K_4$-minors).

Theorem (4-cycle)

If $d_2 = d_4 = 2$, then $I_{\Gamma,d}$ generated in degree 2 and 4, for all $d_1, d_3$.

Theorem

If $\Gamma$ is the boundary of a bipyramid over a simplex of dimension $d$, and all $d_i = 2$, $I_{\Gamma,d}$ generated in degrees $2^{d+1}, 2^d, 2$. 
Conditional Independence Ideals

- \( X = (X_1, \ldots, X_n) \) is an \( n \)-dimensional discrete random vector.
- Probability Distribution
  \[
p_{i_1 \ldots i_n} = \text{Prob}(X_1 = i_1, \ldots, X_n = i_n)
\]
- Let \((A, B, C)\), partition of \([n]\).
- Conditional independence statement \( X_A \indep X_B | X_C \) gives a binomial ideal in \( \mathbb{C}[p_{i_1 \ldots i_n} : i_j \in [r_j]] \).

**Example**

Let \( n = 4, r_1 = r_2 = r_3 = r_4 = 2 \), \( X_1 \indep X_3 | (X_2, X_4) \), gives CI ideal

\[
I_{X_1 \indep X_3 | (X_2, X_4)} = \langle p_{1111}p_{2121} - p_{1121}p_{2111}, p_{1112}p_{2122} - p_{1122}p_{2112}, p_{1211}p_{2221} - p_{1221}p_{2211}, p_{1212}p_{2222} - p_{1222}p_{2212} \rangle
\]
Global Conditional Independence Ideals

- $G$ undirected graph, vertex set $[n]$.
- CI statement $X_A \perp \perp X_B | X_C$ holds for $G$ if $C$ separates $A$ from $B$ in $G$.
- Let $\text{Global}(G)$ set of all CI statements holding for $G$.

$$I_{\text{Global}(G)} = \sum_{X_A \perp \perp X_B | X_C \in \text{Global}(G)} I_{X_A \perp \perp X_B | X_C}$$

**Example (4-cycle)**

$$I_{\text{Global}(G)} = I_{X_1 \perp \perp X_3 | (X_2, X_4)} + I_{X_2 \perp \perp X_4 | (X_1, X_3)}$$
Proposition

If $G = G_1 \# G_2$ is a clique sum, then

$$I_{\text{Global}}(G) = I_{\text{Global}}(G_1) \times A I_{\text{Global}}(G_2)$$

and $A$ is linearly independent.

Example

For $r_i = 2$ for all $i$, $I_{\text{Global}}(G)$ has $9 \times 9 = 81$ prime components.

Conjecture

For all graphs $G$, $I_{\text{Global}}(G)$ is a radical ideal.

All nontoric components are related to marginal positivity.
Problems in Algebraic Statistics call for the study of ideals associated to graphs and simplicial complexes.

Graph theory provides decomposition theory.

Algebraic structure of ideals reflected in structure of graphs (conjecturally).

Toric fiber product is an algebraic decomposition tool for proving results for some graph classes.

References

- A. Engström and T. Kahle. Multigraded commutative algebra of graph decompositions. 1102.2601