Combinatorial Properties of Hierarchical Models

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**Definition**

Let $C \subseteq 2^m$ be simplicial complex and $S \subseteq [m]$. Let $\nu_S \in \mathbb{N}^C$ be the vector with

$$\nu_S(T) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

Let $A_C = \{ \nu_S : S \subseteq [m] \} \subseteq \mathbb{N}^C$ be the vector configuration.

**Example**

Let $C = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\} = 1 \ 2 \ 3$.

\[
A_C = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
Let $\mathcal{A} = \{v_1, \ldots, v_r\} \subseteq \mathbb{N}^d$.

$\mathbb{N}\mathcal{A} := \left\{ \sum_{i=1}^r \lambda_i v_i : \lambda_1, \ldots, \lambda_r \in \mathbb{N} \right\}$ (Semigroup generated by $\mathcal{A}$)

$\mathbb{Z}\mathcal{A} := \left\{ \sum_{i=1}^r \lambda_i v_i : \lambda_1, \ldots, \lambda_r \in \mathbb{Z} \right\}$ (Lattice generated by $\mathcal{A}$)

$\mathbb{R}_{\geq 0}\mathcal{A} := \left\{ \sum_{i=1}^r \lambda_i v_i : \lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0} \right\}$ (Cone generated by $\mathcal{A}$)

**Definition**

$\mathcal{A} \in \mathbb{N}^d$ is called **normal** if

$$\mathbb{N}\mathcal{A} = \mathbb{Z}\mathcal{A} \cap \mathbb{R}_{\geq 0}\mathcal{A}.$$  

$\mathcal{A} \in \mathbb{N}^d$ is called **unimodular** if for all $b \in \mathbb{Z}\mathcal{A} \cap \mathbb{R}_{\geq 0}\mathcal{A}$ the polytope

$$P(\mathcal{A}, b) := \{ x \in \mathbb{R}^r_{\geq 0} : Ax = b \}$$

has all integral vertices.
Main Problem and Motivation

Problem

Classify simplicial complexes $\mathcal{C}$ such that $A_{\mathcal{C}}$ is normal or unimodular.

- Terminology abuse: “$\mathcal{C}$ is normal” or “$\mathcal{C}$ is unimodular”

- If $\mathcal{C}$ is unimodular, associated integer programs are easy to solve.

$$\min / \max \ p_\emptyset \quad \text{s.t.} \quad A_{\mathcal{C}} p = b, \ p \in \mathbb{N}^{2m}.$$ 

- Normality of $\mathcal{C}$ related to integer table feasibility problem in combinatorial optimization.

- Sequential importance sampling works best with normal $\mathcal{C}$.

- Toric fiber product construction for computing Markov bases works best when $\mathcal{C}$ is normal [Rauh-Sullivant 2014].
Some Prior Results on Normality

Theorem (Sullivant 2010, Ohsugi 2010)

If $C$ is a graph it is normal if and only if it is free of $K_4$ minors.

Theorem (Bruns, Hibi, Ohsugi, et al 2007-2011)

Let $C = \partial \Delta_{m-1}$. Then $A_{C,d}$ is normal in precisely the following situations up to symmetry:

1. At most two of the $d_v$ are greater than two
2. $m = 3$ and $d = (3, 3, a)$ for any $a \in \mathbb{N}$
3. $m = 3$ and $d = (3, 4, 4), (3, 4, 5)$ or $(3, 5, 5)$.

Theorem (Rauh-Sullivant 2014)

Let $C$ be the four-cycle graph. Then $A_{C,d}$ is normal if $d = (2, a, 2, b)$ or $d = (2, a, 3, b)$ with $a, b \in \mathbb{N}$. 
Proposition

If $C$ is a normal simplicial complex then so is:

- vertex deletion: $C \setminus v$
- face contraction: $C / F$
- link of a vertex: $\text{link}_v(C)$
- addition of a cone vertex: $\text{cone}_v(C)$

If $C_1$ and $C_2$ are normal simplicial complexes then so is the complex $C_1 \# C_2$ obtained by gluing along a common face.

Proposition

If $C$ is a normal complex and $F \in C$ is a facet satisfying the $0, \pm 1$ inequality condition then $C \setminus \{F\}$ is normal.
Computational Results: Thanks Winfried and Christof!

- **≤ 3 vertices:** All complexes are normal.
- **4 vertices:** All but two complexes normal.

- **5 vertices:** Eliminating all complexes whose normality/nonnormality follows from restriction to 4 vertex complexes leaves 14 complexes. Normality checked using Normaliz [Bruns et al]
- **6 vertices:** Eliminating all complexes whose normality/nonnormality follows from restriction to 5 vertex complexes leaves 80 complexes. Normality checked using Normaliz [Bruns et al] using development version.

Problem

- Develop new procedures for constructing normal $C$.
- Develop methods for creating holes in $\mathbb{NA}_C$. 
Proposition

If $C$ is a unimodular simplicial complex then so is the:

- **vertex deletion**: $C \setminus v$
- **face contraction**: $C / F$
- **vertex link**: $\text{link}_v(C)$
- **Alexander dual**: $C^*$
- **addition of a cone vertex**: $\text{cone}_v(C)$
- **Lawrence lifting**: $\Lambda(C)$

Proposition

- $\Delta_m \sqcup \Delta_n$ is unimodular. ($\implies$ graphic matroid)
- $(\Delta_m \sqcup \Delta_n)^*$ is unimodular.
Definition
A simplicial complex $\Gamma$ is a minor of $\mathcal{C}$ if it can be obtained from $\mathcal{C}$ by performing a sequence of vertex deletions and vertex links.

Corollary
If $\Delta$ is unimodular than so is any minor of $\Delta$.

Proposition
The following are minor-minimal non-unimodular complexes:
- Union of a point and boundary of a simplex: $\Delta_0 \sqcup \partial \Delta_m$, $m \geq 1$.
- The boundary of the octahedron $O_6$ and its dual $O_6^*$.
- $P_4$, $A_1$, $A_1^*$, $A_2$: 
Theorem (Bernstein-Sullivant 2015)

Let $\mathcal{C}$ be a simplicial complex. The following conditions are equivalent:

1. $\mathcal{C}$ is unimodular.
2. (β-avoiding) $\mathcal{C}$ has no minor isomorphic to $\Delta_0 \sqcup \partial \Delta_m$, $m \geq 1$, $O_6$, $O_6^*$, $P_4$, $A_1$, $A_1^*$, and $A_2$.
3. (nuclear) $\mathcal{C}$ can be built up starting from $\Delta_m \sqcup \Delta_n$ or $\Delta_m$ using the operations:
   - Alexander duality
   - Adding cone vertices
   - Lawrence lifting
Proof Outline

- **Nuclear $\implies$ unimodular**
  - $\Delta_m$ and $\Delta_m \sqcup \Delta_n$ are unimodular.
  - Alexander dual, Lawrence lifting, and cone vertices preserve unimodularity.

- **Unimodular $\implies$ $\beta$-avoiding**
  - Unimodularity preserved under taking minors
  - All $\beta$-complexes are not unimodular

- **$\beta$-avoiding $\implies$ nuclear**
  - Characterize 1-skeleton of $\beta$-avoiding complexes
  - Using induction:
    - Let $C$ be $\beta$-avoiding on $m$ vertices.
    - $\text{link}_v(C)$ and $C \setminus v$ are $\beta$-avoiding $= \text{nuclear}$ on $m - 1$ vertices.
    - Analyzes cases for nuclear $\text{link}_v(C)$ and $C \setminus v$ to see which ones yield $\beta$-avoiding complexes.
    - Show results are nuclear.
Given a simplicial complex $C$ on $[n]$ and an integer vector $d = (d_1, \ldots, d_n)$ with $d_i \geq 2$, is $A_{C,d}$ unimodular?

**Corollary (Bernstein-Sullivant 2015)**

If $A_{C,d}$ is unimodular, then $C$ is nuclear.

Let $C$ and $d$ be specified by the figure below. For which values of $p$ and $q$ is $A_{C,d}$ unimodular?
References


Johannes Rauh and Seth Sullivant. Lifting markov bases and higher codimension toric fiber products ArXiv:1404.6392 2014
