Orthogonality - Worksheet

**Part One**

1. What is the cosine of the angle between \( x = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and \( y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)?

2. Are vectors \( v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \) and \( v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \) orthogonal? How do you know?

3. What are the two conditions necessary for a collection of vectors to be orthonormal?

4. Compute the coordinates of the point \( a = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \) in the basis \( \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \). Draw a picture which shows the point \( a \) and the new basis vectors on a coordinate plane. Imagine that you had many data points. Can you see how using the new coordinates in this non-orthonormal basis would provide us with a completely different (and distorted) view of the data?
Part Two

1. Let

\[ U = \frac{1}{3} \begin{pmatrix} -1 & 2 & 0 & -2 \\ 2 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ -2 & 1 & 0 & 2 \end{pmatrix} \]

a. Show that \( U \) is an orthogonal matrix.

b. Let \( b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \). Solve the equation \( Ux = b \).

2. Find two vectors which are orthogonal to \( x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \).

3. Draw the orthogonal projection of the points onto the subspace \( \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \).

List of Key Words.

- cosine
- orthogonal
- orthonormal
- orthogonal Matrix
- orthonormal basis
- orthogonal projection