Review Packet 1

1. For each of the following, write the vector or matrix that is specified:
   a. \( \mathbf{e}_3 \in \mathbb{R}^4 \)
   b. \( \mathbf{D} = \text{diag}\{2, \sqrt{3}, -1\} \)
   c. \( \mathbf{e} \in \mathbb{R}^3 \)
   d. \( \mathbf{I}_2 \)

2. For each of the following matrices and vectors, give their dimension. Label each as a matrix or vector. For each matrix, indicate whether the matrix is square or rectangular.
   a. \[ \mathbf{A} = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \text{ 3x3 square matrix} \]
   b. \[ \mathbf{h} = \begin{pmatrix} -1 \\ -4 \\ 1 \\ 2 \end{pmatrix} \text{ 4x1 vector} \]
   c. \[ \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \\ B_{41} & B_{42} & B_{43} \end{pmatrix} \text{ 4x3 rectangular matrix} \]
   d. \[ \mathbf{A} = [A_{ij}] \text{ where } i = 1, 2, 3 \text{ and } j = 1, 2 \text{ 3x2 rectangular matrix} \]

3. Specify whether the following augmented matrices are in row-echelon form (REF), reduced row-echelon form (RREF), or neither:
   a. \[ \begin{pmatrix} 3 & 2 & 1 & | & 2 \\ 0 & 2 & 0 & | & 1 \\ 0 & 0 & 1 & | & 5 \end{pmatrix} \text{ REF (could eliminate (1,2) -element and (1,3) -element)} \]
   b. \[ \begin{pmatrix} 3 & 2 & 1 & | & 2 \\ 0 & 2 & 0 & | & 1 \\ 0 & 4 & 0 & | & 0 \end{pmatrix} \text{ neither} \]
   c. \[ \begin{pmatrix} 1 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ RREF} \]
   d. \[ \begin{pmatrix} 1 & 2 & 0 & | & 2 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \text{ REF} \]
4. (True/False) The normal equations are used to find the ordinary least-squares solution to an inconsistent system of equations.  

**TRUE**.

5. If the matrix equation $\mathbf{Mv} = \mathbf{b}$ is inconsistent, what alternative equation should I solve to find a solution $\hat{\mathbf{v}}$ such that $\mathbf{M}\hat{\mathbf{v}} = \mathbf{b}$ is as close to $\mathbf{b}$ as possible in the sense that it minimizes the sum of squared error:

$$
SSE = \sum_{i=1}^{n} (b_i - \hat{b}_i)^2
$$

$$
\mathbf{M}^T\mathbf{M}\hat{\mathbf{v}} = \mathbf{M}^T\mathbf{b} \quad (\text{the normal eqns})
$$

6. Answer the following questions about each matrix

$$
\mathbf{A} = \begin{pmatrix} 4 & 5 & 2 \\ 5 & 3 & 1 \\ 2 & 1 & -2 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}
$$

a. Is the matrix square?  

A **YES**  B **NO**

b. What is the transpose of the matrix?

$$
\mathbf{A}^T = \mathbf{A} \quad \mathbf{B}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}
$$

c. Is the matrix symmetric?

A **YES**  B **NO**

d. If possible, name the diagonal elements of the matrix.

A **4, 3, -2**  B **n, p,**

e. If possible, compute the Trace of the matrix.

A **4 + 3 - 2**  B **n, p,**

f. Can the product $\mathbf{AB}$ be computed? If so, what is the size of the result?

**YES**  **3 x 4**

g. Can the product $\mathbf{BA}$ be computed? If so, what is the size of the result?

**NO**

h. Can the product $(\mathbf{B}_{3\times3})^T(\mathbf{A}_{3\times3})^T$ be computed? If so, what is the result?

**YES**  $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$
7. What is the inverse of the matrix $D = \sigma I_3$?

$$D^{-1} = \frac{1}{\sigma} I_3 = \begin{pmatrix} \frac{1}{\sigma} & 0 & 0 \\ 0 & \frac{1}{\sigma} & 0 \\ 0 & 0 & \frac{1}{\sigma} \end{pmatrix}$$

8. For the following graph, number the nodes and write the corresponding adjacency matrix:

![Graph with nodes and connections]

**Answer**: Depends on order of nodes

9. Compute the outer product $xy^T$ where

$$x = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$xy^T = \begin{pmatrix} 3 & 3 & -3 & 3 \\ 4 & 4 & -4 & 4 \\ 5 & 5 & -5 & 5 \end{pmatrix}$$

10. Can you say anything about the rank of an outer product in general? Explain your answer.

   It will always equal 1 b/c the rows will be multiples of each other \( \Rightarrow \) one linearly ind.

11. Briefly explain what it means for a matrix to be full rank.

   Either the rows are linearly independent or the columns are linearly independent.

   For an $A_{mxn}$ then \( \text{rank}(A) = \min(m,n) \)

12. For the following augmented matrices, circle the pivot elements and give the rank of the coefficient matrix along with the number of free variables.

   a. \[
   \begin{pmatrix}
   3 & 2 & 1 & 1 & 2 \\
   0 & 2 & 0 & 0 & 1 \\
   0 & 0 & 1 & 0 & 5 \\
   \end{pmatrix}
   \]

   rank = 3  # free var = 1
b. \[
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}
\] \quad \text{rank}=2 \quad \# \text{ free var}=1

c. \[
\begin{pmatrix}
1 & 2 & 0 & 1 & 0 & 2 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\] \quad \text{rank}=3 \quad \# \text{ free var}=2

13. Write the vector \( \mathbf{v} \) as a linear combination of each given \( \mathbf{x} \) and \( \mathbf{y} \), if possible.

\[
\mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}
\]

a. \( \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathbf{v} = 2 \mathbf{x} + 3 \mathbf{y} \\

b. \( \mathbf{x} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \mathbf{v} = -2 \mathbf{x} - 3 \mathbf{y} \\

c. \( \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{not possible} \quad \mathbf{v} = a \mathbf{x} + b \mathbf{y} \Leftrightarrow \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{system is inconsistent}

14. Suppose we measure the heights of 10 people, \( \text{person}_1, \text{person}_2, \ldots, \text{person}_{10} \).

a. If we create a matrix \( \mathbf{S} \) where

\[
S_{ij} = \text{height}(\text{person}_i) - \text{height}(\text{person}_j)
\]

is the matrix \( \mathbf{S} \) symmetric? What is the trace(\( \mathbf{S} \))?

\[ \text{No, } S_{ji} = -S_{ij} \text{ so not symmetric. Trace}(\mathbf{S}) = 0 \]

Since \( S_{ii} = 0 \) for all \( i \).

b. If instead we create a matrix \( \mathbf{G} \) where

\[
G_{ij} = [\text{height}(\text{person}_i) - \text{height}(\text{person}_j)]^2
\]

is the matrix \( \mathbf{G} \) symmetric? What is the trace(\( \mathbf{G} \))?

\[ \text{Yes, } G_{ij} = G_{ji} \Rightarrow \text{symmetric. Trace}(\mathbf{G}) = 0 \]

15. For the matrix \( \mathbf{H} = \mathbf{X}(\mathbf{X}^T \mathbf{\Sigma}^{-1} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{\Sigma}^{-1} \), use the properties of matrix arithmetic to show that

a. \( \mathbf{H}^2 = \mathbf{H} \)

\[ \mathbf{H}^2 = \mathbf{H} \quad \mathbf{H} = \mathbf{X}(\mathbf{X}^T \mathbf{\Sigma}^{-1} \mathbf{X})^{-1} \mathbf{X}^T \mathbf{\Sigma}^{-1} \mathbf{X} \]

b. \( \mathbf{H}(\mathbf{I} - \mathbf{H}) = 0 \)

\[ \mathbf{H}(\mathbf{I} - \mathbf{H}) = 0 \quad \text{by part a} \]

16. Let
\[ U = (U_1 | U_2 | U_3 | \ldots | U_p) \] and \[ V^T = \begin{pmatrix} V_1^T \\ V_2^T \\ V_3^T \\ \vdots \\ V_p^T \end{pmatrix} \]

Write the matrix product $UV^T$ in terms of the columns of $U$ and the rows of $V^T$.
\[ UV^T = U_1 V_1^T + U_2 V_2^T + U_3 V_3^T + \ldots + U_p V_p^T \]

17. Suppose that $u$ is a unit vector. Then, $\|u\|_2$ = ?

1. That’s the def. of a unit vector!

18. Let $x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $y = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$. Compute the following:

a. $\|x\|_2 = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$

b. $\|y\|_1 = |0| + |-1| + |-3| = 4$

c. $\|y\|_\infty = \max \{ |0| , |-1| , |-3| \} = 3$

d. $\|x - y\|_2 = \sqrt{(1-0)^2 + (-1+1)^2 + (2+3)^2} = \sqrt{26}$

e. $\|x - y\|_1 = |1-0| + |-1+1| + |2+3| = 6$

19. Suppose we have a dataset containing survey data. Individuals were asked to respond ‘yes’=1 or ‘no’=0 to twenty potential political referendums. Let $a$ be the vector containing the numerical responses of Individual A and let $b$ be the vector containing the numerical responses of Individual B (so $a, b \in \mathbb{R}^{20}$). Explain in words the interpretation of the quantity

$\|a - b\|_1$.

the number of referendums upon which Individ. A and Indiv. B did not agree.
20. **Statistical Formulas Using Linear Algebra Notation.** Almost every statistical formula can be written in a more compact fashion using linear algebra. Most of the elementary formulas involve vector inner products or the Euclidean norm. To begin, we’ll introduce the concept of centering the data. **Centering** the data means that the mean of a variable is subtracted from each observation. For example, if we have some variable, \( \mathbf{x} \), and 3 observations on that variable:

\[
\mathbf{x} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}
\]

then obviously, \( \bar{x} = 3 \). The **centered** version of \( \mathbf{x} \) would then be

\[
\mathbf{x} - \bar{x}\mathbf{e} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.
\]

We simply subtract the mean from every observation so that the new mean of the variable is 0.

Most multivariate textbooks start by saying “all variable vectors in this textbook are assumed to be centered to have mean zero unless otherwise specified”. Looking at the most common statistical formulas helps us see why. Try to re-write the following formulas using linear algebra notation, using the vectors \( \mathbf{x} \) and \( \mathbf{y} \) to represent centered data:

\[
\mathbf{x} = \begin{pmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ x_3 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 - \bar{y} \\ y_2 - \bar{y} \\ y_3 - \bar{y} \\ \vdots \\ y_n - \bar{y} \end{pmatrix}
\]

For this exercise, keep in mind the following linear algebra constructs, which you should be very familiar with by now:

\[
\| \mathbf{a} \| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2}
\]

\[
\mathbf{a}^T \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n
\]

a. Sample standard deviation:

\[
s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}} = \sqrt[2]{\frac{\| \mathbf{x} \|}{n-1}}
\]

b. Sample covariance:

\[
covariance(\mathbf{x}, \mathbf{y}) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n-1} \mathbf{x}^\top \mathbf{y}
\]
c. Correlation coefficient:

\[
 r_{xy} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2 \sum_{i=1}^{n} (y_i - \bar{y})^2}} = \frac{x^T y}{\|x\| \cdot \|y\|} \\
\]

Note: this is just the inner product of unit vectors!

\[
\left(\frac{x}{\|x\|}\right)^T \left(\frac{y}{\|y\|}\right)
\]