Worksheet - Lecture 14
Partitioned Matrix Arithmetic

1. Suppose I want to compute the matrix product $A = UDV^T$ where $U$ is $n \times r$, $D$ is an $r \times r$ diagonal matrix, $D = diag\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$, and $V^T$ is $r \times p$. (Side note: we will quite often want to compute such a matrix product – this is the form of the singular value decomposition (SVD)! The following exercise is not just for fun - what you end up with in part b is exactly how we will want to write the SVD to best understand how it works.)

a. Using what you know about multiplication by diagonal matrices, if we view the matrix $U$ as a collection of columns,

$U = (U_1 | U_2 | U_3 | \ldots | U_r)$

then how would I write the same partition of the matrix $UD$?

$UD = (?)| (?)| (?)| \ldots | (?)$

Diagonal Matrix on Right simply scales the columns of $U$ by diagonal elements

$UD = (\sigma_1 U_1 | \sigma_2 U_2 | \sigma_3 U_3 | \ldots | \sigma_r U_r)$

Keep in mind that when multiplying matrices/vectors by scalars, it is always preferable to write the scalar first (ex: $x \sigma$ rather than $\sigma x$)

b. Now, using the above representation for $UD$, what happens when I multiply by the matrix $V^T$, viewed as a collection of rows,

$V^T = \begin{pmatrix} V_1^T \\ V_2^T \\ \vdots \\ V_r^T \end{pmatrix}$

$UDV^T = ?$

$(\sigma_1 U_1 | \sigma_2 U_2 | \ldots | \sigma_r U_r) \begin{pmatrix} V_1^T \\ V_2^T \\ \vdots \\ V_r^T \end{pmatrix} = \begin{pmatrix} \sigma_1 U_1 V_1^T + \sigma_2 U_2 V_2^T + \ldots + \sigma_r U_r V_r^T \end{pmatrix}$

(Hint: your answer should be a sum. Each term in the sum should be an outer product.)
2. Consider

\[
A = \begin{pmatrix}
-1 & 2 & 4 & 1 & 0 \\
1 & 0 & -1 & -2 & 1 \\
2 & -1 & 3 & 1 & 2 \\
1 & 2 & 3 & 4 & 3 \\
-1 & -2 & 0 & 1 & 2
\end{pmatrix}
\quad \text{and} \quad
x = \begin{pmatrix}
1 \\
2 \\
3 \\
4 \\
5
\end{pmatrix}
\]

Partition these into submatrices (regions/block) conformably for multiplication as follows:

\[
A = \begin{pmatrix}
A_{00} & A_{01} & A_{02} \\
A_{10}^T & a_{11} & a_{12} \\
A_{20} & A_{21} & A_{22}
\end{pmatrix}
\quad \text{and} \quad
x = \begin{pmatrix}
x_0 \\
x_1 \\
x_2
\end{pmatrix}
\]

Where \(A_{00}\) is a \(3 \times 3\) matrix, \(x_0 \in \mathbb{R}^3\), \(a_{11}\) is a scalar and \(x_1\) is a scalar. Show with lines how \(A\) and \(x\) are partitioned below:

\[
A = \begin{pmatrix}
-1 & 2 & 4 & 1 & 0 \\
1 & 0 & -1 & -2 & 1 \\
2 & -1 & 3 & 1 & 2 \\
1 & 2 & 3 & 4 & 3 \\
-1 & -2 & 0 & 1 & 2
\end{pmatrix}
\quad \text{and} \quad
x = \begin{pmatrix}
1 \\
2 \\
3 \\
4 \\
5
\end{pmatrix}
\]

3. For all matrices \(A_{n \times k}\) and \(B_{k \times n}\), show that the block matrix

\[
L = \begin{pmatrix}
I - BA & B \\
2A - ABA & AB - I
\end{pmatrix}
\begin{pmatrix}
I - BA & B \\
2A - ABA & AB - I
\end{pmatrix}
\]

satisfies the property \(L^2 = I\). Hint: Perform block matrix multiplication for each of the four separate blocks in the result, simplifying each expression as much as possible.

\[
(1,1) - \text{Block} : (I - BA)(I - BA) + B(2A - ABA) = I - 2BA + BABABA + 2BA - BABABA = I
\]

\[
(1,2) - \text{Block} : (I - BA)B + B(AB - I) = B - BAB + BAB - B = 0
\]

\[
(2,1) - \text{Block} : (2A - ABA)(I - BA) + (AB - I)(2A - ABA) = 2A - ABA - 2ABA + ABABA + 2ABA - ABABA - 2A + ABA = 0
\]

\[
(2,2) - \text{Block} : (2A - ABA)B + (AB - I)(AB - I) = 2AB - ABAB + ABAB - 2AB + I = I
\]

So, Final result \(\begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix} = I\)