Chapter 6
Linear Independence
A set of vectors \( \{v_1, v_2, \ldots, v_p \} \) is **linearly dependent** if we can express the zero vector, \( 0 \), as a *non-trivial* linear combination of the vectors.

\[
\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p = 0
\]

(non-trivial means that all of the \( \alpha_i \)'s are not 0).
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\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p = \mathbf{0}
\]

(non-trivial means that all of the \( \alpha_i \)'s are not 0).

The set \( \{v_1, v_2, \ldots, v_p\} \) is **linearly independent** if the above equation has only the trivial solution, \( \alpha_1 = \alpha_2 = \cdots = \alpha_p = 0 \).
Linear Dependence - Example

The vectors \( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \), \( \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \), and \( \mathbf{v}_3 = \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix} \) are linearly dependent because

\[ \mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2 \]
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$$v_3 = 2v_1 + v_2$$

or, equivalently, because

$$2v_1 + v_2 - v_3 = 0$$
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# PerfectMulticollinearity!
Example - Determining Linear Independence

\( \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \) and \( \mathbf{v}_3 = \begin{pmatrix} 3 \\ 6 \\ 7 \end{pmatrix} \)

How can we tell if these vectors are linearly independent?

- Want to know if there are coefficients \( \alpha_1, \alpha_2, \alpha_3 \) such that

\[
\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = 0
\]

- This creates a linear system!

\[
\begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 2 & 3 & 7 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

- Just use Gauss-Jordan elimination to find out that

\[
\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}
\]

is one possible solution (there are free variables)!
For a set of vectors \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \),

- If the only solution was the trivial solution,
  \[
  \begin{pmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \alpha_3 
  \end{pmatrix}
  =
  \begin{pmatrix}
  0 \\
  0 \\
  0 
  \end{pmatrix}
  \]
  Then we’d know that \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are linearly independent.
- \( \implies \) no free variables! Gauss-Jordan elimination on the vectors results in the identity matrix:
  \[
  \begin{pmatrix}
  1 & 0 & 0 & | & 0 \\
  0 & 1 & 0 & | & 0 \\
  0 & 0 & 1 & | & 0 
  \end{pmatrix}
  \]
The sum from our definition,

\[ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p = 0, \]

is simply a matrix-vector product

\[ V\alpha = 0 \]

where \( V = (v_1 | v_2 | \ldots | v_p) \) and \( \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{pmatrix} \).
The sum from our definition,

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p = 0,$$

is simply a matrix-vector product

$$V\alpha = 0$$

where $V = (v_1 \mid v_2 \mid \cdots \mid v_p)$ and $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{pmatrix}$

So all we need to do is determine whether the system of equations $V\alpha = 0$ has any non-trivial solutions.
If a set of vectors (think: *variables*) is not linearly independent, then the matrix that contains those vectors as columns (think: *data matrix*) is not full rank!

The **rank** of a matrix can be defined as the number of linearly independent columns (or rows) in that matrix.

- **# of linearly independent rows = # of linearly independent columns**

In most data - **# of rows > # of columns**.

So the maximum rank of a matrix is the **# of columns** - an \( n \times m \) full rank matrix has \( rank = m \).
Linear Independence

Let $A$ be an $n \times n$ matrix. The following statements are equivalent. (If one of these statements is true, then all of these statements are true)

- $A$ is invertible ($A^{-1}$ exists)
- $A$ has full rank ($\text{rank}(A) = n$)
- The columns of $A$ are linearly independent
- The rows of $A$ are linearly independent
- The system $Ax = b$ has a unique solution
- $Ax = 0 \Rightarrow x = 0$
- $A$ is nonsingular

$\xrightarrow{\text{Gauss-Jordan}} I$
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- The rows of $A$ are linearly independent
- The system $Ax = b$ has a unique solution
- $Ax = 0 \implies x = 0$
- $A$ is nonsingular
- $A$ Gauss–Jordan $\rightarrow I$
Let \( \mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \) and \( \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \).

- Are the vectors \( \mathbf{a} \) and \( \mathbf{b} \) linearly independent?
Let \( \mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \) and \( \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \).

- Are the vectors \( \mathbf{a} \) and \( \mathbf{b} \) linearly independent?

- What is the rank of the matrix \( \mathbf{A} = (\mathbf{a} | \mathbf{b}) \)?
Check your understanding

Let \( \mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \) and \( \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \).

- Are the vectors \( \mathbf{a} \) and \( \mathbf{b} \) linearly independent?

- What is the rank of the matrix \( \mathbf{A} = (\mathbf{a}|\mathbf{b}) \)?

- Determine whether or not the vector \( \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \) is a linear combination of the vectors \( \mathbf{a} \) and \( \mathbf{b} \).
Let \( \mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \) and \( \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \).

- Are the vectors \( \mathbf{a} \) and \( \mathbf{b} \) linearly independent?
  
  Yes. The equation \( \alpha_1 \mathbf{a} + \alpha_2 \mathbf{b} = \mathbf{0} \) has only the trivial solution.

What is the rank of the matrix \( \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \)? Is \( \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \) full rank?

The rank of \( \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \) is 2 because there are two linearly independent columns. 

\( \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \) is full rank.

Determine whether or not the vector \( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \) is a linear combination of the vectors \( \mathbf{a} \) and \( \mathbf{b} \).

Row reduce the augmented matrix:

\[
\begin{pmatrix}
1 & 3 & 1 \\
3 & 0 & 0 \\
4 & 1 & 1 \\
\end{pmatrix}
\]

to find that the system is inconsistent. 

\( \Rightarrow \) No.
Let \( \mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \) and \( \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \).

- Are the vectors \( \mathbf{a} \) and \( \mathbf{b} \) linearly independent?
  Yes. The equation \( \alpha_1 \mathbf{a} + \alpha_2 \mathbf{b} = \mathbf{0} \) has only the trivial solution.

- What is the rank of the matrix \( \mathbf{A} = (\mathbf{a} | \mathbf{b}) \)? Is \( \mathbf{A} \) full rank?
  \( \text{rank}(\mathbf{A}) = 2 \) because there are two linearly independent columns. \( \mathbf{A} \) is full rank.
Let \( \mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \) and \( \mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \).

- Are the vectors \( \mathbf{a} \) and \( \mathbf{b} \) linearly independent?
  Yes. The equation \( \alpha_1 \mathbf{a} + \alpha_2 \mathbf{b} = \mathbf{0} \) has only the trivial solution.

- What is the rank of the matrix \( \mathbf{A} = (\mathbf{a} \, | \, \mathbf{b}) \)? Is \( \mathbf{A} \) full rank?
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- Determine whether or not the vector \( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \) is a linear combination of the vectors \( \mathbf{a} \) and \( \mathbf{b} \).
  Row reduce the augmented matrix:
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  \begin{pmatrix}
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  to find that the system is inconsistent. \( \implies \) No.
Why the fuss?

If our design matrix $\mathbf{X}$ is not full rank, then the matrix from the normal equations, $\mathbf{X}^T\mathbf{X}$ is also not full rank.

- $\mathbf{X}^T\mathbf{X}$ does not have an inverse.
- The normal equations do not have a unique solution!
- $\beta$’s not uniquely determined.
- Infinitely many solutions.
- #PerfectMulticollinearity
- Breaks a fundamental assumption of MLR.
Often times we’ll run into a situation where variables are linearly independent, but only barely so. Take, for example, the following system of equations:

\[ \beta_1 x_1 + \beta_2 x_2 = y \]

where

\[ x_1 = \begin{pmatrix} 0.835 \\ 0.333 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0.667 \\ 0.266 \end{pmatrix}, \quad y = \begin{pmatrix} 0.168 \\ 0.067 \end{pmatrix} \]

This system has an exact solution, \( \beta_1 = 1 \) and \( \beta_2 = -1 \).
\[ \beta_1 x_1 + \beta_2 x_2 = y \]

where

\[
x_1 = \begin{pmatrix} 0.835 \\ 0.333 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0.667 \\ 0.266 \end{pmatrix} \quad y = \begin{pmatrix} 0.168 \\ 0.067 \end{pmatrix}
\]

If we change this system only slightly, so that \( y = \begin{pmatrix} 0.168 \\ 0.066 \end{pmatrix} \) then

the exact solution changes drastically to

\[ \beta_1 = -666 \quad \text{and} \quad \beta_2 = 834 \]

The system is **unstable** because the columns of the matrix are so close to being linearly dependent!
Symptoms of Severe Multicollinearity

- Large fluctuations or flips in sign of the coefficients when a collinear variable is added into the model.
- Changes in significance when additional variables are added.
- Overall F-test shows significance when the individual t-tests show none.

These symptoms are bad enough on their own, but the real consequence of this type of behavior is that seen in the previous example. A very small change in the underlying system of equations (like a minuscule change in a target value $y_i$) can produce dramatic changes to our parameter estimates!