1. Suppose I want to compute the matrix product \( \mathbf{A} = \mathbf{U} \mathbf{D} \mathbf{V}^T \) where \( \mathbf{U} \) is \( n \times r \), \( \mathbf{D} \) is an \( r \times r \) diagonal matrix, \( \mathbf{D} = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_r\} \), and \( \mathbf{V}^T \) is \( r \times p \). (Side note: we will quite often want to compute such a matrix product – this is the form of the singular value decomposition (SVD)! The following exercise is not just for fun - what you end up with in part b is exactly how we will want to write the SVD to best understand how it works.)

a. Using what you know about multiplication by diagonal matrices, if we view the matrix \( \mathbf{U} \) as a collection of columns,

\[
\mathbf{U} = (\mathbf{U}_1 | \mathbf{U}_2 | \mathbf{U}_3 | \ldots | \mathbf{U}_r)
\]

then how would I write the same partition of the matrix \( \mathbf{U} \mathbf{D} \)?

\[
\mathbf{U} \mathbf{D} = (?|?|?|\ldots|?)
\]

Keep in mind that when multiplying matrices/vectors by scalars, it is always preferable to write the scalar first (\( \sigma \mathbf{x} \) rather than \( \mathbf{x} \sigma \))

b. Now, using the above representation for \( \mathbf{U} \mathbf{D} \), what happens when I multiply by the matrix \( \mathbf{V}^T \), viewed as a collection of rows,

\[
\mathbf{V}^T = \begin{pmatrix}
\mathbf{V}_1^T \\
\mathbf{V}_2^T \\
\mathbf{V}_3^T \\
\vdots \\
\mathbf{V}_r^T
\end{pmatrix}
\]

\[
\mathbf{U} \mathbf{D} \mathbf{V}^T = ?
\]

(Hint: your answer should be a sum. Each term in the sum should be an outer product.)
2. Consider

\[
A = \begin{pmatrix}
-1 & 2 & 4 & 1 & 0 \\
1 & 0 & -1 & -2 & 1 \\
2 & -1 & 3 & 1 & 2 \\
1 & 2 & 3 & 4 & 3 \\
-1 & -2 & 0 & 1 & 2 \\
\end{pmatrix}
\quad \text{and} \quad
x = \begin{pmatrix}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{pmatrix}
\]

Partition these into submatrices (regions/blocks) conformably for multiplication as follows:

\[
A = \begin{pmatrix}
A_{00} & a_{01} & A_{02} \\
a_{10}^T & a_{11} & a_{12}^T \\
A_{20} & a_{21} & A_{22} \\
\end{pmatrix}
\quad \text{and} \quad
x = \begin{pmatrix}
x_0 \\
x_1 \\
x_2 \\
\end{pmatrix}
\]

Where \(A_{00}\) is a 3 \times 3 matrix, \(x_0 \in \mathbb{R}^3\), \(a_{11}\) is a scalar and \(x_1\) is a scalar. Show with lines how A and x are partitioned below:

\[
A = \begin{pmatrix}
-1 & 2 & 4 & 1 & 0 \\
1 & 0 & -1 & -2 & 1 \\
2 & -1 & 3 & 1 & 2 \\
1 & 2 & 3 & 4 & 3 \\
-1 & -2 & 0 & 1 & 2 \\
\end{pmatrix}
\quad \text{and} \quad
x = \begin{pmatrix}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{pmatrix}
\]

3. For all matrices \(A_{n \times k}\) and \(B_{k \times n}\), show that the block matrix

\[
L = \begin{pmatrix}
I - BA & B \\
2A - ABA & AB - I \\
\end{pmatrix}
\]

satisfies the property \(L^2 = I\). Hint: Perform block matrix multiplication for each of the four separate blocks in the result, simplifying each expression as much as possible.