Homework Due March 1 - answers

Ch. 6, Problem 1: Find an isomorphism from the group of integers under addition to the group of even integers under addition.

Solution: I will show that the map \( f(x) = 2x \) is an isomorphism. Any even integer is of the form \( 2n \) for some integer \( n \), so \( f \) is onto. If \( f(x) = f(y) \), then \( 2x = 2y \) so \( x = y \). Therefore \( f \) is one-to-one. Since \( f(x + y) = 2(x + y) = 2x + 2y \) we can now conclude that \( f \) is an isomorphism.

Ch. 6, Problem 4: Show that \( U(8) \) is not isomorphic to \( U(10) \).

Solution: Looking at the multiplication table for \( U(8) \) one sees that \( x^2 = 1 \) for all \( x \in U(8) \). In \( U(10) \) we have \( 3 \cdot 3 = 9 \neq 1 \). If \( f : U(8) \to U(10) \) were an isomorphism and \( f(a) = 3 \), we would have \( 1 = f(1) = f(a \cdot a) = f(a) \cdot f(a) = 3 \cdot 3 = 9 \), a contradiction.

Ch. 6, Problem 5: Show \( U(8) \) is isomorphic to \( U(12) \).

Solution: The multiplication tables are, respectively:

\[
\begin{array}{ccc|ccc}
U(8) & 1 & 3 & 5 & 7 \\
1 & 1 & 3 & 5 & 7 \\
3 & 3 & 1 & 7 & 5 \\
5 & 5 & 7 & 1 & 3 \\
7 & 7 & 5 & 3 & 1 \\
\end{array}
\quad
\begin{array}{ccc|ccc}
U(12) & 1 & 5 & 7 & 11 \\
1 & 1 & 5 & 7 & 11 \\
5 & 5 & 1 & 11 & 7 \\
7 & 7 & 11 & 1 & 5 \\
11 & 11 & 7 & 5 & 1 \\
\end{array}
\]

The map \( f : U(8) \to U(12) \) defined by \( f(1) = 1, f(3) = 5, f(5) = 7, f(7) = 11 \) is an isomorphism because it is one-to-one, onto and preserves the group laws by inspection.

Ch. 6, Problem 10: Let \( G \) be a group. Prove that the mapping \( \alpha(g) = g^{-1} \) for all \( g \) in \( G \) is an automorphism if and only if \( G \) is Abelian.

Solution: The mapping is always one-to-one because \( g^{-1} = h^{-1} \) implies that \( g = (g^{-1})^{-1} = (h^{-1})^{-1} = h \). The mapping is always onto because for and \( g \in G \), \( \alpha(g^{-1}) = g \).

Assume that \( \alpha \) is an isomorphism. We then have that \( (ab)^{-1} = a^{-1}b^{-1} \) for all \( a, b \in G \). Therefore \( b^{-1}a^{-1} = a^{-1}b^{-1} \). Taking inverses of both sides, we have \( ab = ba \) so \( G \) is abelian. Conversely, if \( ab = ba \) for all \( a, b \in G \), then \( b^{-1}a^{-1} = a^{-1}b^{-1} \). This gives the statement that \( \alpha(ab) = \alpha(a)\alpha(b) \).