Chapter 6 Solutions

1. \( u(x,t) = e^{-4t} \sin x - 3e^{-100t} \sin 5x \).

2. \( u(x,t) = (2 \cos 3\pi t + \frac{2}{3} \sin 3\pi t) \sin \pi x + 7 \cos 9\pi t \sin 3\pi x \).

3. \( \lambda = 0 \) is an eigenvalue if and only if \( a_0 + L\alpha a_L + a_L = 0 \) with eigenfunction \( u(x) = a_0 x + 1 \).

4. a) Set \( \lambda = -k^2 \). Then \( k \) satisfies \( \tanh kL = k|a_0 + a_L|/(k^2 + a_0 a_L) := f(k) \). This is maximized at \( k = \sqrt{a_0 a_L} \) with \( f(k^*) \geq 1 \), so it must intersect \( \tanh kL \) at some value \( k > 0 \).
   b) If \( f'(0) < L \) then \( f \) crosses \( \tanh kL \) twice to produce two eigenvalues (which is also sufficient). This is equivalent to \( a_0 + a_L + L\alpha a_L > 0 \).

5. a) Formalize the graphical arguments in the section.
   b) \( \beta_n = \tan^{-1}(f(\beta_n)) + (n - 1)\pi/L \) but \( f(\beta_n) \to 0 \) as \( n \to \infty \) so \( \theta_n := \beta_n - (n - 1)\pi/L \to 0 \).
   c) Note that \( f(Cn + \theta_n) = (a_0 + a_L)/(Cn) + O(1/n^2) \) and \( \tan(1/n) = 1/n + O(1/n^2) \), which gives that \( \theta_n = (a_0 + a_L)/\pi n + O(1/n^2) \).

6. There are two eigenvalues if \( a_0 + a_L + L\alpha a_L > 0 \), one if it equals zero and none otherwise. Neumann boundary conditions correspond to \( a_0 = a_L = 0 \) and Dirichlet to \( a_0 = a_L = \pm \infty \).

7. Set \( \alpha_i = \sqrt{r_i/p_i} \) and \( \lambda = k^2 \). Continuity of \( u \) and \( u' \) at \( x = m \) gives two equations, along with two more for the boundary conditions. With \( S_i = \sin(k_s i m) \) and \( C_i = \cos(k_s i m) \), a non-trivial solution exists when \( \tan(k\alpha_2 L) = (\alpha_2 S_2 + \alpha_1 C_1 C_2)/(-\alpha_2 S_1 C_2 + \alpha_1 C_1 S_2) \).

8. a) Look for values of \( \mu > 0 \) such that \( \cos \mu L = -1/cos \mu L \). Note that \( \cos \mu L \) has zeros at \( \mu L = n\pi + \pi/2 \) and has minima at \( \mu L = \pi + 2n\pi \); solutions exist between minima and adjacent zeros.
   b) Set \( L\mu_n = (n - 1/2)\pi + \theta_j \). Then \( \cos(L\mu_n) \sim -2e^{-\mu_n L} \) implies \( \theta_n \sim 2(-1)^{n+1}e^{\pi/2}e^{-\pi n} \).

9. a) Use (6.5) to obtain \( b_n = 4/\pi n \) for \( n \) odd and zero otherwise.
   b) Evaluating the series at \( x = \pi/4 \) gives \( 1 + 1/3 - 1/5 - 1/7 + 1/9 + \cdots = \frac{\pi}{4} \sqrt{2} \).