8. Consider the linear transport equation (1.8) with initial and boundary conditions (1.10).

(a) Suppose the data \( \phi, \psi \) are differentiable functions. Show that the function \( u : Q_1 \to \mathbb{R} \) given by

\[
    u(x, t) = \begin{cases} 
        \phi(x - ct), & \text{if } x \geq ct, \\
        \psi(t - x/c), & \text{if } x \leq ct
    \end{cases}
\]  

(1.13)

satisfies the PDE away from the line \( x = ct \), the boundary condition, and initial condition. To see where (1.13) comes from, start from the general solution \( u(x, t) = f(x - ct) \) of the PDE and substitute into the side conditions (1.10).

(b) In solution (1.13), the line \( x = ct \), which emerges from the origin \( x = t = 0 \), separates the quadrant \( Q_1 \) into two regions. On the line, the solution has one-sided limits given by \( \phi, \psi \). Consequently, the solution will in general have singularities on the line.

   (i) Find conditions on the data \( \phi, \psi \) so that the solution is continuous across the line \( x = ct \).

   (ii) Find conditions on the data \( \phi, \psi \) so that the solution is differentiable across the line \( x = ct \).

9. Let \( f : \mathbb{R} \to \mathbb{R} \) be differentiable. Verify that if \( u(x, t) \) is differentiable and satisfies (1.12), that is, \( u = f(x - ut) \), then \( u(x, t) \) is a solution of the initial value problem

\[
    u_t + uu_x = 0, \quad -\infty < x < \infty, \quad t > 0, \quad u(x, 0) = f(x), \quad -\infty < x < \infty.
\]

10. Let \( u_0(x) = 1 - x^2 \) if \(-1 \leq x \leq 1\), and \( u_0(x) = 0 \) otherwise.

(a) Use (1.12) to find a formula for the solution \( u = u(x, t) \) of the inviscid Burgers equation (1.11), (1.9) with \(-1 < x < 1, 0 < t < \frac{1}{2}\).

(b) Verify that \( u(1, t) = 0, \quad 0 < t < \frac{1}{2} \).

(c) Differentiate your formula to find \( u_x(1^-, t) \), and deduce that \( u_x(1^-, t) \to -\infty \) as \( t \to \frac{1}{2}^- \).

Note: \( u_x(x, t) \) is discontinuous at \( x = \pm 1 \); the notation \( u_x(1^-, t) \) means the one-sided limit: \( u_x(1^-, t) = \lim_{x \to 1^-} u_x(x, t) \). Similarly, \( t \to \frac{1}{2}^- \) means \( t \to \frac{1}{2} \), with \( t < \frac{1}{2} \).
Figure 1.2. Inviscid Burgers equation: nonlinear wave propagation. (a) \( t = 0 \); (b) \( t > 0 \).

Eventually, the graph becomes infinitely steep, and the implicit solution in (1.12) is no longer valid. The solution is continued to larger time by including a shock wave, defined in Chapter 13.

PROBLEMS

1. Show that the traveling wave \( u(x, t) = f(x - 3t) \) satisfies the linear transport equation \( u_t + 3u_x = 0 \) for any differentiable function \( f : \mathbb{R} \to \mathbb{R} \).

2. Find an equation relating the parameters \( k, m, n \) so that the function \( u(x, t) = e^{mt} \sin(nx) \) satisfies the heat equation \( u_t = ku_{xx} \).

3. Find an equation relating the parameters \( c, m, n \) so that the function \( u(x, t) = \sin(mt) \sin(nx) \) satisfies the wave equation \( u_{tt} = c^2 u_{xx} \).

4. Find all functions \( a, b, c : \mathbb{R} \to \mathbb{R} \) such that \( u(x, t) = a(t)e^{2x} + b(t)e^x + c(t) \) satisfies the heat equation \( u_t = u_{xx} \) for all \( x, t \).

5. For \( m > 1 \), define the conductivity \( k = k(u) \) so that the porous medium equation (1.7) can be written as the (quasilinear) heat equation

\[
u_t = \nabla \cdot (k(u) \nabla u).\]

6. Solve the initial value problem

\[
\begin{align*}
    u_t + 4u_x &= 0, & -\infty < x < \infty, & t > 0, \\
    u(x, 0) &= (1 + x^2)^{-1}, & -\infty < x < \infty.
\end{align*}
\]

7. Solve the initial boundary value problem

\[
\begin{align*}
    u_t + 4u_x &= 0, & 0 < x < \infty, & t > 0, \\
    u(x, 0) &= 0, & 0 < x < \infty, \\
    u(0, t) &= te^{-t}, & t > 0.
\end{align*}
\]

Explain why there is no solution if the PDE is changed to \( u_t - 4u_x = 0 \).