1. (25 points) Solve the initial value problem for \( u = u(x, t) \):

\[
\begin{align*}
ut + (x + e^t) u_x - 1 &= 0, \quad -\infty < x < \infty, \ t > 0 \\
u(x, 0) &= 1/(1 + x^2), \quad -\infty < x < \infty.
\end{align*}
\]

**Solution:** \( u(x, t) = t + 1/(1 + (xe^{-t} - t)^2) \)

2. (20 points) Find the regions in the \((x, y)\)-plane where the equation

\[
xu_{yy} - yu_{xy} + u_{xx} + 2u_x - u_y + 2u = x + y
\]

is elliptic/hyperbolic/parabolic. Draw a diagram to display your answer.

**Solution:** \( y^2 > 4x \): Hyperbolic; \( y^2 < 4x \): Elliptic; \( y^2 = 4x \): Parabolic.

3. (55 points) Consider d’Alembert’s solution of the Cauchy problem for the wave equation. Let \( u(x, t) \) be the solution of the initial value problem

\[
\begin{align*}
utt &= 4u_{xx}, \quad -\infty < x < \infty, \ t > 0 \\
u(x, 0) &= 0, \quad -\infty < x < \infty, \\
u_t(x, 0) &= \begin{cases} 
1 - |x|, & \text{if } |x| \leq 1 \\
0 & \text{if } |x| > 1.
\end{cases}
\end{align*}
\]

(a) Sketch the \(x, t\)-plane showing significant characteristics.

\[
\text{Figure 1: The } x, t \text{ plane with characteristics, showing the line with } 1/4 < t < 1/2.
\]
(b) Let $t$ satisfy $\frac{1}{4} < t < \frac{1}{2}$. Write a formula for $u(x, t)$, $x \geq 0$, evaluating all integrals involved.

Solution: $u(x, t) = \frac{1}{8}(1 + (x - 2t)^2) - \frac{1}{4}(x - 2t)$ in region a: $2t < x < 2t + 1$

$u(x, t) = -\frac{1}{4}(x - 2t) - \frac{1}{8}((x - 2t)^2 - 1)$ in region b: $1 - 2t < x < 2t$

$u(x, t) = -\frac{1}{4}(x - 2t) - \frac{1}{8}((x - 2t)^2 + (x + 2t)^2) + \frac{1}{4}(x + 2t)$ in region c: $0 < x < 1 - 2t$

$u = 0$ for $x > 2t + 1$.

(c) Draw a graph of $u(x, t)$ as a function of $x \geq 0$, where $\frac{1}{4} < t < \frac{1}{2}$.

![Graph of $u(x, t)$ with $1/4 < t < 1/2$ fixed and $x > 0$.]