The entry-exit function
and geometric singular perturbation theory

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The model

\[ u_t = D_u u_{xx} + u(1 - u) - \frac{u}{\alpha + u} v, \]

\[ v_t = D_v v_{xx} + \delta v \left(1 - \frac{\beta}{u} v \right). \]

- \( u \) = prey, \( v \) = predator.
- Both populations are subject to overcrowding.
- Predator carrying capacity is proportional to prey population.
- For fixed predator population, prey is consumed at a rate that stabilizes as its population increases.
Traveling waves


- Look for traveling waves with velocity \( c > 0 \), set \( z = x - ct \).
- Rescale space so \( c = 1 \).
- Set \( \epsilon = \frac{D_u}{c^2} \), \( \mu = \frac{D_v}{D_u} \). (Small \( \epsilon > 0 \) means small diffusion.)

Traveling waves \((u, v)(z)\) satisfy:

\[
0 = \epsilon u_{zz} + u_z + u(1 - u) - \frac{u}{\alpha + u} v,
\]

\[
0 = \epsilon \mu v_{zz} + v_z + \delta v \left(1 - \frac{\beta}{u}\right).
\]

Rewrite as a first-order system.
A slow-fast system in slow form

\[
\begin{align*}
    u_z &= U, \\
    \epsilon U_z &= -U - u(1 - u) + \frac{u}{\alpha + u} v, \\
    v_z &= V, \\
    \epsilon \mu V_z &= -V - \delta v \left(1 - \frac{\beta}{u} v\right).
\end{align*}
\]

Normally attracting critical manifold (set \( \epsilon = 0 \)):

\[
\begin{align*}
    U &= -u(1 - u) + \frac{u}{\alpha + u} v, \\
    V &= -\delta v \left(1 - \frac{\beta}{u} v\right).
\end{align*}
\]

Slow system:

\[
\begin{align*}
    u_z &= \frac{u}{\alpha + u} v - u(1 - u), \\
    v_z &= \delta v \left(\frac{\beta}{u} v - 1\right).
\end{align*}
\]

Undefined for \( u = 0 \). Multiply by \((\alpha + u)u\):
Another slow-fast system, in fast form

\[
\begin{align*}
\dot{u} &= u^2 (v - (1 - u)(\alpha + u)), \\
\dot{v} &= \delta v (\alpha + u) (\beta v - u).
\end{align*}
\]

Small \( \delta > 0 \) means slowly changing predator population. \( u = 0 \) is now invariant. **Note the factor** \( u^2 \).

For small \( \delta > 0 \), numerical simulation shows a closed orbit near a “singular orbit” with a certain value of \( v_0^* \).
Classical analog

\[ \dot{x} = \epsilon f(x, z), \]
\[ \dot{z} = g(x, z)z, \]

with \( x \in \mathbb{R}, z \in \mathbb{R}, \)

\[ f(x, 0) > 0, \quad g(x, 0) \text{ has the sign of } x. \]

- **Note the factor** \( z. \)
- For \( \epsilon = 0, \) the \( x \)-axis consists of equilibria.
- Normally attracting for \( x < 0, \) normally repelling for \( x > 0. \) **Loss of normal hyperbolicity at** \( z = 0. \)

- For \( \epsilon > 0, \) \( x \)-axis remains invariant, flow is to the right.
Entry-exit function: attraction and repulsion balance

For small $\epsilon > 0$, a solution that starts at $(x_0, z_0)$, with $x_0$ negative and $z_0 > 0$ small, reintersects the line $z = z_0$ at $(p_\epsilon(x_0), z_0)$.

**Theorem**

As $\epsilon \to 0$, $p_\epsilon(x_0) \to p_0(x_0)$ given implicitly by

$$\int_{x_0}^{p_0(x_0)} \frac{g(x, 0)}{f(x, 0)} \, dx = 0.$$

The solution leaves the $x$-axis when repulsion has built up to balance the attraction that occurred before $x = 0$. 
If the theorem holds when \( z \) is replaced by \( z^2 \), how to prove existence of the closed orbit

- Define \( v_0^* \) by \( \int_{v_0^*}^{v_1^*} \frac{v - \alpha}{\alpha \beta v^2} \, dv = 0 \).
- Follow the flow backwards for small \( \delta > 0 \).
- \( p_\delta : \Sigma_1 \to \Sigma_0 \) (entry-exit function) would be smooth.
- \( q_\delta : \Sigma_0 \to \Sigma_1 \) is an exponential contraction.
- \( q_\delta \circ p_\delta : \Sigma_1 \to \Sigma_1 \) has a fixed point.
Reformulation

\[ \dot{x} = \epsilon f(x, z), \]
\[ \dot{z} = g(x, z)z. \]

\[ f(x, 0) > 0, \quad g(x, 0) \text{ has the sign of } x. \]

Divide by \( f(x, z) > 0 \), let \( h = g/f \):

\[ \dot{x} = \epsilon, \]
\[ \dot{z} = h(x, z)z. \]

\[ h(x, 0) \text{ has the sign of } x. \]

Theorem

As \( \epsilon \to 0 \), \( p_\epsilon(x_0) \to p_0(x_0) \) given implicitly by

\[ \int_{x_0}^{p_0(x_0)} h(x, 0) \, dx = 0. \]
“Standard” proofs that $p_\epsilon(x_0) \to p_0(x_0)$


- Comparison to solutions constructed by separation of variables: S., J. Diff. Eq. 60 (1985), 131–141.


De Maesschalck: Let $p(x_0, \epsilon) = p_\epsilon(x_0)$, $\epsilon \geq 0$. If $f$ and $g$ are $C^r$, $r \geq 1$, then $p$ is $C^r$. 
How to prove $p_\epsilon(x_0) \to p_0(x_0)$ in the $C^0$ sense

\[ \dot{x} = \epsilon, \]
\[ \dot{z} = h(x, z)z. \]

$h(x, 0)$ has the sign of $x$.

Replace by

\[ \dot{x} = \epsilon, \]
\[ \dot{z} = (h(x, 0) \pm \alpha)z. \]

\[ \frac{dz}{dx} = \frac{(h(x, 0) \pm \alpha)z}{\epsilon} \quad \Rightarrow \quad \frac{\epsilon}{z} dz = (h(x, 0) \pm \alpha)dx. \]

If a solution that starts on the line $z = z_0$ at $x = x_0$ reintersects it when $x = x_1$, then

\[ 0 = \int_{x_0}^{x_1} (h(x, 0) \pm \alpha) \, dx = 0. \]
- De Maesschalck’s proof doesn’t seem to work for $z$ replaced by $z^2$.
- Proofs don’t use the blow-up method of Geometric Singular Perturbation Theory, today the usual approach to loss of normal hyperbolicity in slow-fast systems.
De Maesschalck and S., J. Diff. Eq. 260 (2016), 6697–6715

\[ \dot{x} = \epsilon, \]
\[ \dot{z} = h(x, z)z^2. \]

\( h(x, 0) \) has the sign of \( x \).

Using blow-up, we prove:

**Theorem**

If \( h \) is \( C^\infty \), then:

1. There is a \( C^\infty \) function \( \tilde{p} \) of three variables such that \( p(x_0, \epsilon) = \tilde{p}(x_0, \epsilon, \epsilon \log \epsilon) \).

2. If \( h(x, z) - h(x, 0) \) is \( C^\infty \) flat in \( z \), then \( p \) is a \( C^\infty \) function of \( (x_0, \epsilon) \).
A change of variables and the classical situation

\[ \dot{x} = \epsilon, \]
\[ \dot{z} = h(x, z)z. \]

\[ z = \kappa(w) = \begin{cases} e^{-\frac{1}{w}} & \text{if } w > 0, \\ 0 & \text{if } w = 0. \end{cases} \]

\[ \dot{x} = \epsilon, \]
\[ \dot{w} = h(x, \kappa(w))w^2, \]

because

\[ \dot{z} = \kappa'(w)\dot{w} = e^{-\frac{1}{w}} \frac{1}{w^2} \dot{w} \Rightarrow \]
\[ \dot{w} = e^{\frac{1}{w}} \dot{z}w^2 = e^{\frac{1}{w}} h(x, \kappa(w))e^{-\frac{1}{w}} w^2. \]

Notice \( h(x, \kappa(w)) - h(x, 0) \) is \( C^\infty \) flat in \( w \).
Classical result recovered

\[ \begin{align*}
\dot{x} &= \epsilon, \\
\dot{z} &= h(x, z)z.
\end{align*} \]

\( h(x, 0) \) has the sign of \( x \).

**Theorem**

*If \( h \) is \( C^\infty \), then \( p(x_0, \epsilon) \) is \( C^\infty \).*

This result is not new, but it follows from part 2 of the Main Theorem.

Thus a linear result is a consequence of a quadratic result.
Proof of Main Theorem: Extension of the system

\[
\begin{align*}
\dot{x} &= \epsilon, \\
\dot{z} &= h(x, z)z^2, \\
\dot{\epsilon} &= 0.
\end{align*}
\]

\(h(x, 0)\) has the sign of \(x\).

Define \(P : R_0 \to R_3\) by \(P(x_0, \epsilon) = (p_\epsilon(x_0), \epsilon)\), with \(p_0\) defined implicitly by \(\int_{x_0}^{p_0(x_0)} h(x, 0) \, dx = 0\). Study \(P\).
Blow-up transformation

\[ \dot{x} = \epsilon, \]
\[ \dot{z} = h(x, z)z^2, \]
\[ \dot{\epsilon} = 0. \]

Let \((x, (\bar{z}, \bar{\epsilon}), r)\) be a point of \(\mathbb{R} \times S^1 \times \mathbb{R}_+; \bar{z}^2 + \bar{\epsilon}^2 = 1\).

Blow-up transformation:

\[ x = x, \quad z = r\bar{z}, \quad \epsilon = r\bar{\epsilon}. \]

Our system pulls back to one on \(\mathbb{R} \times S^1 \times \mathbb{R}_+\).

Division by \(r\) desingularizes the new system on the cylinder \(r = 0\) but leaves it invariant.
Cylindrical coordinates

The blow-up can be visualized most completely in cylindrical coordinates.

For \((x, (\bar{z}, \bar{\epsilon}), r) \in \mathbb{R} \times S^1 \times \mathbb{R}_+\), let \(\bar{z} = \cos \theta\) and \(\bar{\epsilon} = \sin \theta\).

\[
\begin{align*}
x &= x, \\
z &= r \cos \theta, \\
\epsilon &= r \sin \theta.
\end{align*}
\]

After making the coordinate change and dividing by \(r\), the system becomes

\[
\begin{align*}
\dot{x} &= \sin \theta, \\
\dot{r} &= r \cos^3 \theta \cdot h(x, r \cos \theta), \\
\dot{\theta} &= -\cos^2 \theta \sin \theta \cdot h(x, r \cos \theta).
\end{align*}
\]
Flow in cylindrical coordinates

\[ \dot{x} = \sin \theta, \]
\[ \dot{r} = r \cos^3 \theta \, h(x, r \cos \theta), \]
\[ \dot{\theta} = -\cos^2 \theta \sin \theta \, h(x, r \cos \theta). \]
Affine coordinates

\[\begin{align*}
\dot{x} &= \epsilon, \\
\dot{z} &= h(x, z)z^2, \\
\dot{\epsilon} &= 0.
\end{align*}\]

New coordinates that blow up the \(x\)-axis to a plane:

\[\begin{align*}
x &= x, \\
z &= z, \\
\epsilon &= zE.
\end{align*}\]

The plane \(z = 0\) in \(xzE\)-space corresponds to the line \(z = \epsilon = 0\) in \(xz\epsilon\)-space.

Change variables, divide by \(z\) (otherwise the plane \(z = 0\) is all equilibria):

\[\begin{align*}
\dot{x} &= E, \\
\dot{z} &= h(x, z)z, \\
\dot{E} &= -h(x, z)E.
\end{align*}\]
Flow in affine coordinates

\[ \dot{x} = E, \]
\[ \dot{z} = h(x, z)z, \]
\[ \dot{E} = -h(x, z)E. \]

Notice \( \epsilon = zE \) is a first integral.

0 = \int_{x_0}^{x_3} \frac{dE}{dx} dx = \int_{x_0}^{x_3} -h(x, 0) dx \Rightarrow x_3 = p_0(x_0).
Return map as a composition

\[ P = P_3 \circ P_2 \circ P_1. \]

\( P_2 \) is clearly \( C^\infty \).

It remains to study the smoothness of \( P_1 \) and \( P_3 \).
Simplification

\[ \dot{x} = E, \]
\[ \dot{z} = h(x, z)z, \]
\[ \dot{E} = -h(x, z)E \]

At the left, divide by \(-h(x, z) > 0\), let \(k = -\frac{1}{h} > 0\):

\[ \dot{x} = k(x, z)E, \]
\[ \dot{z} = -z, \]
\[ \dot{E} = E. \]

Note that \(zE = \epsilon\) is constant on solutions.
Normal form step 1

\[ \dot{x} = k(x, z)E, \]
\[ \dot{z} = -z, \]
\[ \dot{E} = E. \]

Straighten flow on \( z = 0 \):

Then \( \dot{x} = \tilde{k}(\bar{x}, z, E)zE = \epsilon \tilde{k}(\bar{x}, z, E). \)
Normal form step 2

Proposition

Let $N \geq 1$. Then after a $C^\infty$ coordinate change

$$\bar{x} = \eta(x, z, E),$$

$$\dot{x} = \epsilon a(\bar{x}, \epsilon) + \epsilon^N b(\bar{x}, z, E),$$

$$\dot{z} = -z,$$

$$\dot{E} = E,$$

with $a$ and $b$ of class $C^\infty$.

Proof: The case $N = 1$ was step 1 (with $a = 0$). If the proposition is true for some $N$, let

$$\hat{x} = \bar{x} + \epsilon^N (\beta(\bar{x}, z) + \gamma(\bar{x}, E)),$$

and choose $\beta$ and $\gamma$ to eliminate terms of order $\epsilon^N$. 
Integration change of variables

\[
\begin{align*}
\dot{x} &= \epsilon a(x, \epsilon) + \epsilon^{N+2} b(x, z, E), \\
\dot{z} &= -z, \\
\dot{E} &= E,
\end{align*}
\]

Integrate from \((\bar{x}_0, 1, \epsilon)\) to \((\bar{x}_1, \frac{\epsilon}{E_1}, E_1)\). Change of variables:

\[
\bar{z} = zE \log z = \epsilon \log z, \quad \bar{E} = zE \log E = \epsilon \log E, \quad \tau = \frac{t}{\epsilon}.
\]

Use \(\dot{\bar{z}} = \frac{\epsilon}{\bar{z}} \dot{z} = -\epsilon\), etc.:

\[
\begin{align*}
\bar{x}' &= a(\bar{x}, \epsilon) + \epsilon^{N+1} b(\bar{x}, e^{\bar{z}/\epsilon}, e^{\bar{E}/\epsilon}), \\
\bar{z}' &= -1, \\
\bar{E}' &= 1.
\end{align*}
\]

Integrate from \((\bar{x}_0, 0, \epsilon \log \epsilon)\) to \((\bar{x}_1, \epsilon \log \frac{\epsilon}{E_1}, \epsilon \log E_1)\).
Region of integration

\[ \begin{align*}
\ddot{x}' &= a(\bar{x}, \epsilon) + \epsilon^{N+1} b(\bar{x}, e^{\bar{z}/\epsilon}, e^{\bar{E}/\epsilon}), \\
\ddot{z}' &= -1, \\
\ddot{E}' &= 1.
\end{align*} \]

Regard \( \epsilon \) as a parameter. Integrate from \((\bar{x}_0, 0, \epsilon \log \epsilon)\) to \((\bar{x}_1, \epsilon \log \frac{\epsilon}{E_1}, \epsilon \log E_1)\).

Within the region of integration \( D \), the system is \( C^N \).
Smoothness

\[ \bar{x}' = a(\bar{x}, \epsilon) + \epsilon^{N+1} b(\bar{x}, e^{\bar{z}/\epsilon}, e^{\bar{E}/\epsilon}) \]

\[ \bar{z} = \epsilon \log (\epsilon/E_1) \quad \bar{E} = \epsilon \log E_1 \]

\[ \frac{\partial^N}{\partial \epsilon^N} \epsilon^{N+1} b(\bar{x}, e^{\bar{z}/\epsilon}, e^{\bar{E}/\epsilon}) \]

\[ = \epsilon^{N+1} D_3 b(\bar{x}, e^{\bar{z}/\epsilon}, e^{\bar{E}/\epsilon}) \left( -\frac{\bar{E}}{\epsilon^2} \right)^N + \ldots \]

\[ = (-1)^N \frac{\bar{E}^N}{\epsilon^{N-1}} D_3 b(\bar{x}, e^{\bar{z}/\epsilon}, e^{\bar{E}/\epsilon}) + \ldots. \]

Within \( \mathcal{D} \), \( \frac{\bar{E}^N}{\epsilon^{N-1}} \rightarrow 0 \) as \((\bar{z}, \bar{E}, \epsilon) \rightarrow (0, 0, 0)\).
Output depends on \((\bar{x}_0, \varepsilon, \varepsilon \log \varepsilon)\)

\[
\begin{align*}
\bar{z} &= \varepsilon \log \left(\frac{\varepsilon}{E_1}\right) & \bar{E} &= \varepsilon \log E_1 \\
\bar{x} &= \phi(\bar{x}_0, \bar{E}_0, \varepsilon, \tau), \text{ where } \phi \text{ is } C^N \text{ as long as the solution remains in } \mathcal{D}. \text{ Thus} \nonumber \\
\bar{x}_1 &= \phi(\bar{x}_0, \bar{E}_0, \varepsilon, \tau) = \phi(\bar{x}_0, \varepsilon \log \varepsilon, \varepsilon, \varepsilon \log E_1 - \varepsilon \log \varepsilon). 
\end{align*}
\]

More compactly, \(\bar{x}_1\) is a \(C^N\) function of \((\bar{x}_0, \varepsilon, \varepsilon \log \varepsilon)\).
Exchange Lemma

\[ \dot{x} = \epsilon f(x, z, \epsilon), \]
\[ \dot{z} = g(x, z, \epsilon)z, \]
\[ (x, z) \in \mathbb{R}^n \times \mathbb{R}, \quad f(x, 0, 0) \neq 0, \quad g(x, 0, 0) < 0. \]

Theorem

If \( N_0 \) and the \( N_\epsilon \) (\( \epsilon > 0 \)) fit together to form a smooth manifold of \( xz\epsilon \)-space, then, away from \( P_0, P_0^* \) and the \( N_\epsilon^* \) (\( \epsilon > 0 \)) do too.
Ting-Hao Hsu, preprint, 2016

Slow-fast system #1

\[
\begin{align*}
\dot{x} &= \epsilon, \\
\dot{z} &= h(x, z)z,
\end{align*}
\]

\(h(x, 0)\) has the sign of \(x\).

Dot indicates derivative with respect to fast time \(t\).

Introduce the slow time \(\tau = \epsilon t\) as a new dependent variable.

Slow-fast system #2

\[
\begin{align*}
\dot{x} &= \epsilon, \\
\dot{z} &= h(x, z)z, \\
\dot{\tau} &= \epsilon.
\end{align*}
\]
Slow-fast system #2

\[ \dot{x} = \epsilon, \]
\[ \dot{z} = h(x, z)z, \]
\[ \dot{\tau} = \epsilon. \]

For \( x_0 < 0 \):
- Define \( x_1 \) by \( \int_{x_0}^{x_1} h(x, 0) \, dx = 0. \)
- Define \( \tau_1 = x_1 - x_0. \)
Slow-fast system #2

\[ \dot{x} = \epsilon, \]
\[ \dot{z} = h(x, z)z, \]
\[ \dot{\tau} = \epsilon. \]

Start and end 1-dimensional manifolds:

- Define \( N_\epsilon : (x, z, \tau) = (x_0, z_0, \sigma) : |\sigma| < \delta. \)
- Define \( Q_\epsilon : (x, z, \tau) = (x_1 + \sigma, z_0, \tau_1) : |\sigma| < \delta. \)

Idea: For \( \epsilon > 0 \), start at \((x, z) = (x_0, z_0)\), return to \( z = z_0 \):
will have \( x \approx x_1 \) and \( \Delta \tau \approx \tau_1 \).
Project $N_0$ and $Q_0$ along the fast flow to $z = 0$. Get $P_0$ and $R_0$.

Follow $P_0$ and $R_0$ using the slow system on $z = 0$.

Obtain $P_0^*$ and $R_0^*$, both open subsets of $x\tau$-space.

This does not help us study the intersection of $N_\varepsilon^*$ and $Q_\varepsilon^*$, which are near $P_0^*$ and $R_0^*$. 
Introduce an “extra variable” using $z = e^{-\frac{\zeta}{\epsilon}}$ or:

$$\zeta = -\epsilon \ln z,$$

so

$$\dot{\zeta} = -\epsilon \frac{\dot{z}}{z} = -\epsilon h(x, z).$$

**Slow-fast system #3 (equivalent to original system on the invariant manifold $\zeta = -\epsilon \ln z$)**

\[\begin{align*}
\dot{x} &= \epsilon, \\
\dot{z} &= h(x, z)z, \\
\dot{\zeta} &= -\epsilon h(x, z), \\
\dot{\tau} &= \epsilon.
\end{align*}\]

New start and end 1-dimensional manifolds:

- $N_\epsilon: (x, z, \tau) = (x_0, z_0, \sigma) : |\sigma| < \delta \rightarrow$
  $\tilde{N}_\epsilon: (x, z, \zeta, \tau) = (x_0, z_0, -\epsilon \ln z_0, \sigma) : |\sigma| < \delta$

- $Q_\epsilon: (x, z, \tau) = (x_1 + \sigma, z_0, \tau_1) : |\sigma| < \delta \rightarrow$
  $\tilde{Q}_\epsilon: (x, z, \zeta, \tau) = (x_1 + \sigma, z_0, -\epsilon \ln z_0, \tau_1) : |\sigma| < \delta$
Project $\tilde{N}_0$ and $\tilde{Q}_0$ along the fast flow to $z = 0$, i.e., to $x\zeta\tau$-space:

- $\tilde{P}_0$: $(x, \zeta, \tau) = (x_0, 0, \sigma): |\sigma| < \delta$.
- $\tilde{R}_0$: $(x, \zeta, \tau) = (x_1 + \sigma, 0, \tau_1): |\sigma| < \delta$.

Follow using the slow system on $z = 0$:

$x' = 1,$
$\zeta' = -h(x, 0),$ 
$\tau' = 1.$
We easily obtain: $\tilde{P}_0^*$ and $\tilde{R}_0^*$ meet transversally at

$$(x, \zeta, \tau) = (0, \int_{x_0}^{0} -h(\xi, 0) \, d\xi, -x_0)$$

$$= (0, \int_{x_1}^{0} -h(\xi, 0) \, d\xi, -x_0).$$

Equality follows from $\int_{x_0}^{x_1} h(\xi, 0) \, d\xi = 0.$
Summary for Slow-Fast System #3 in $xz\zeta\tau$-space

- Start and end 1-dimensional manifolds $\tilde{N}_\epsilon$ and $\tilde{Q}_\epsilon$.
- Project $\tilde{N}_0$ and $\tilde{Q}_0$ along the fast flow to $\tilde{P}_0$ and $\tilde{R}_0$ in $z = 0$, i.e., in $xz\zeta\tau$-space.
- $\tilde{P}_0^*$ and $\tilde{R}_0^*$ meet transversally (2-dimensional manifolds in $\mathbb{R}^3$).
- By the Exchange Lemma, away from $\tilde{P}_0$ and $\tilde{R}_0$, $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ are close to $\tilde{P}_0^*$ and $\tilde{R}_0^*$ respectively.
- **But we cannot conclude that** $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ **meet transversally.**
  1. Two-dimensional manifolds in $\mathbb{R}^4$.
  2. Exchange Lemma can’t follow $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ to $x = 0$. 

![Diagram of Slow-Fast System](image-url)
**Objection 1:** $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ are 2-dimensional manifolds in $\mathbb{R}^4$.

- $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ are close to $\tilde{P}_0^*$ and $\tilde{R}_0^*$, which meet transversally in $x\zeta\tau$-space.
- Project $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ to $x\zeta\tau$-space (ignore small $z$-coordinate). The projections meet transversally there.
- But then $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ intersect, because on these manifolds, $\zeta = -\epsilon \ln z$. 
Objection 2: Exchange Lemma can’t follow $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ to $x = 0$.

Slow-fast system #3

\[
\begin{align*}
\dot{x} &= \epsilon, \\
\dot{z} &= h(x, z)z, \\
\dot{\zeta} &= -\epsilon h(x, z), \\
\dot{\tau} &= \epsilon.
\end{align*}
\]

Within the manifold $\zeta = -\epsilon \ln z$, in which both $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ lie,

\[
\begin{align*}
\dot{x} &= \epsilon, \\
\dot{\zeta} &= -\epsilon h(x, e^{-\zeta/\epsilon}), \\
\dot{\tau} &= \epsilon.
\end{align*}
\]

In the slow time $\tau = \epsilon t$: 
Within the manifold $\zeta = -\epsilon \ln z$, in which both $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ lie,

\[
\begin{align*}
    x' &= 1, \\
    \zeta' &= -h(x, e^{-\frac{\zeta}{\epsilon}}), \\
    \tau' &= 1
\end{align*}
\]

Use this to follow $\tilde{N}_\epsilon^*$ and $\tilde{Q}_\epsilon^*$ to $x = 0$. 
Recent work that uses an “extra variable” to study exponential loss of normal hyperbolicity in geometric singular perturbation problems


\[
\begin{align*}
\dot{x} &= \epsilon e^{-z^{-1}}, \\
\dot{z} &= z^2 \left( x - e^{-z^{-1}} \right).
\end{align*}
\]

\[
q = e^{-z^{-1}} \Rightarrow \dot{q} = e^{-z^{-1}} z^{-2} \dot{z} = q(x - q).
\]

\[
\dot{x} = \epsilon q, \quad \dot{z} = z^2 (x - q), \quad \dot{q} = q(x - q).
\]