Morse Theory for Lagrange Multipliers

\[ \gamma = 0 \]

\[ \text{grad} \gamma \]

\[ \text{grad} f \]

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Outline


(2) Lagrange multipliers with scaled multiplier

(3) The limit $\lambda \to \infty$

(4) The limit $\lambda \to 0$

(5) Etc.

Section 1. Classical Morse Theory


Setting:

- \( M \) = compact manifold of dimension \( n \).
- \( f : M \rightarrow \mathbb{R} \) is a smooth function.
- \( x \) is a critical point if \( df(x) = 0 \).
- \( x \) is a nondegenerate critical point if \( d^2 f(x) \) has \( k \) negative eigenvalues and \( n - k \) positive eigenvalues.
- index \( x = k \).
- \( f \) is a Morse function if all critical points are nondegenerate.
- \( M^a = \{ x \in M : f(x) \leq a \} \).
Fundamental Theorem of Morse Theory.

Suppose $a < b$ are regular values of a Morse function $f$.

- If $f^{-1}[a, b]$ contains no critical point of $f$, then $M^b$ is diffeomorphic to $M^a$.
- If $f^{-1}[a, b]$ contains one nondegenerate critical point of $f$, with index $k$, then $M^b$ has the homotopy type of $M^a$ with a $k$ handle-attached.

**Figure 1. Height function.**

**Figure 2. Adding handles.**
Section 2. Morse-Smale ODEs

Put a metric on $M$, let $f$ be a Morse function, and consider

$$\dot{x} = -\nabla f(x).$$

If $x$ is a critical point of $f$ of index $k$, then $x$ is a hyperbolic equilibrium with $\dim W^u(x) = k$.

The ODE is **Morse-Smale** if the unstable and stable manifolds of different equilibria meet transversally.

![Diagram of upright and tilted tori](image)

(a) Upright torus: not Morse-Smale.  
(b) Tilted torus: Morse-Smale.

**Theorem.** Morse-Smale ODE’s are structurally stable.
Section 3. Morse homology

Let $f : M \rightarrow \mathbb{R}$ be a Morse function with $\dot{x} = -\nabla f(x)$ Morse-Smale.

Define the Morse-Smale-Witten chain complex:

$C_k = \text{free } \mathbb{Z}_2\text{-module generated by the critical points of index } k.$

Boundary operator $d : C_k \rightarrow C_{k-1}$:

$p = \text{critical point } p \text{ of index } k, \ q = \text{critical point of index } k-1.$

$W^u(p) \cap W^s(q) = n(p, q)$ curves.

$$d(p) = \sum_{\text{critical points } q \text{ of index } k-1} n(p, q) \cdot q.$$  

Proposition. $d \circ d = 0.$
The Morse homology of $f$ is the homology of this chain complex.

**Theorem.** The Morse homology of $f$ equals the singular homology of $M$.

(c) Morse-Smale flow.

(d) Chain complex.

**History**

Apparently known to Thom, Smale, and Milnor.

Rediscovered by Edward Witten in *Supersymmetry and Morse Theory* (J. Diff. Geom., 1982) in which the curves correspond to instantons that represent tunneling to remove spurious degeneracies in a perturbation calculation involving the action of the Laplacian on the deRham complex . . .
Section 4. Floer homology

Method of Andreas Floer, series of papers in 1987-89:

(1) Associate with a manifold $M$ an important infinite-dimensional manifold $X$ (e.g., loop space of a symplectic manifold).

(2) Find a natural functional on $X$ (e.g., symplectic action functional associated to a symplectomorphism) and a natural metric on $X$.

(3) Calculate the Morse homology. If you encounter infinite indices, try to define a finite index difference.

(4) Prove something about $M$ (e.g., Arnold’s conjecture on the number of fixed points of a symplectomorphism).

(5) If you can’t do step 4, investigate a simpler Morse homology analog for inspiration.

Motivated by the symplectic vortex equation, we consider …
Lagrange multipliers with scaled multiplier

\( M = \) compact manifold.

\( f : M \rightarrow \mathbb{R} \) is a Morse function.

\( \gamma : M \rightarrow \mathbb{R} \) has 0 as a regular value (so \( \gamma^{-1}(0) \) is a manifold).

Lagrange function:

\[
\mathcal{L} : M \times \mathbb{R} \rightarrow \mathbb{R}, \quad \mathcal{L}(x, \eta) = f(x) + \eta \gamma(x).
\]

Critical point set of \( \mathcal{L} \):

\[
\text{Crit}(\mathcal{L}) = \{(x, \eta) : df(x) + \eta d\gamma(x) = 0, \ \gamma(x) = 0\},
\]

There is a bijection \( \text{Crit}(\mathcal{L}) \simeq \text{Crit}\left( f \mid \gamma^{-1}(0) \right), (x, \eta) \mapsto x. \)

We’ll investigate the morse homology of \( \mathcal{L} \).
$g$ is a Riemannian metric on $M$.

$e$ is standard metric on $\mathbb{R}$.

$g \oplus e$ is a metric on $M \times \mathbb{R}$.

Gradient vector field of $\mathcal{L}$ with respect to $g \oplus e$:

$$\nabla \mathcal{L}(x, \eta) = (\nabla f + \eta \nabla \gamma, \gamma(x)).$$

ODE for the negative gradient flow of $\mathcal{L}$

$$\dot{x} = -(\nabla f(x) + \eta \nabla \gamma(x)), \quad \dot{\eta} = -\gamma(x).$$

Rescale the metric on the second factor: $g \oplus \lambda^{-2} e$. Distance $\lambda \rightarrow$ distance 1.

New negative gradient ODE:

$$\dot{x} = -(\nabla f(x) + \eta \nabla \gamma(x)), \quad \dot{\eta} = -\lambda^2 \gamma(x).$$

**Theorem.** Morse homology is unchanged (for $\lambda$ for which the chain complex $C^\lambda$ is defined).

Homology of $C^\lambda$ is not the homology of $M \times \mathbb{R}$. 
Limit as $\lambda \to 0$

Lagrange function:

$$\mathcal{L}(x, \eta) = f(x) + \eta \gamma(x).$$

Negative gradient flow with rescaled metric, a fast-slow system for small $\lambda$.

\[ \dot{x} = - (\nabla f(x) + \eta \nabla \gamma(x)) , \]
\[ \dot{\eta} = -\lambda^2 \gamma(x) . \]

Set $\lambda = 0$:

\[ \dot{x} = - (\nabla f(x) + \eta \nabla \gamma(x)) , \]
\[ \dot{\eta} = 0 . \]
1. The **slow manifold** is the set of equilibria for $\lambda = 0$:

\[
\begin{align*}
\dot{x} &= - (\nabla f(x) + \eta \nabla \gamma(x)), \\
\dot{\eta} &= 0.
\end{align*}
\]

\[
S_L = \{(x, \eta) : \nabla f(x) + \eta \nabla \gamma(x) = 0\}.
\]

Assume

- $0$ is a regular value of $\nabla f(x) + \eta \nabla \gamma(x)$ (so $S_L$ is a curve).
- $\eta|_{S_L}$ has nondegenerate critical points.

$s_L^{\text{sing}}$ = critical points of $\eta|_{S_L}$. 

\[ \dot{x} = - (\nabla f(x) + \eta \nabla \gamma(x)), \]
\[ \dot{\eta} = 0. \]

2. The fast equation is \( \dot{x} = - (\nabla f(x) + \eta \nabla \gamma(x)) \).

Let \( f_\eta = f + \eta \gamma : M \to \mathbb{R} \) with \( \eta \) regarded as constant.

- \( x \) is a critical point of \( f_\eta \) if and only if \((x, \eta) \in S_L\).

- \( x \) is a degenerate critical point of \( f_\eta \) if and only if \((x, \eta) \in S_{L}^{\text{sing}}\).

- Assume the fast equation is Morse-Smale except at isolated bifurcation values of \( \eta \) where a single degeneracy occurs (degenerate critical point or nontransverse intersection of stable and unstable manifolds).
3. The slow equation:

\[ \dot{x} = - \left( \nabla f(x) + \eta \nabla \gamma(x) \right), \]
\[ \dot{\eta} = -\lambda^2 \gamma(x). \]

Away from critical points of \( \eta \big|_{S_L} \):

- \( S_L \) is parameterized by \( \eta \): \( x(\eta) \).
- \( S_L \) is normally hyperbolic.
- Slow equation: \( \dot{\eta} = -\gamma(x(\eta)) \).
\[ \dot{x} = - (\nabla f(x) + \eta \nabla \gamma(x)), \]
\[ \dot{\eta} = -\lambda^2 \gamma(x). \]

Slow equation: \( \dot{\eta} = -\gamma(x(\eta)). \)

Equilibria of the slow equation are points of \( S_L \) where \( \gamma = 0. \)
- They are hyperbolic.
- They are the points of \( \text{Crit}(L) \), the critical point set of \( L(x, \eta) = f(x) + \eta \gamma(x) \).
- They are the equilibria of the negative gradient flow of \( L \).

Assume:
- If \( (x, \eta) \in S_L \) and \( \gamma(x) = 0 \), \( \eta \) is not a bifurcation value for the fast equation.
\[ \dot{x} = - (\nabla f(x) + \eta \nabla \gamma(x)), \]
\[ \dot{\eta} = -\lambda^2 \gamma(x). \]

Three Morse indices:

- For any \( p = (x, \eta) \in S_L \), \( \text{index}(x, f_\eta) \).
- For \( p = (x, \eta) \in \text{Crit}(L) \), \( \text{index}(p, L) \).
- For \( p = (x, \eta) \in \text{Crit}(L) \), \( \text{index}(x, f | \gamma^{-1}(0)) \).

Relation for \( p = (x, \eta) \in \text{Crit}(L) \):

- \( \text{index}(p, L) = \text{index}(x, f | \gamma^{-1}(0)) + 1 \).
- If \( p \) is a repeller of the slow equation, then \( \text{index}(p, L) = \text{index}(x, f_\eta) + 1 \).
- If \( p \) is an attractor of the slow equation, then \( \text{index}(p, L) = \text{index}(x, f_\eta) \).
Slow-fast orbits connecting $p \in \text{Crit}(\mathcal{L})$ with index $(p, \mathcal{L}) = k$ to $q \in \text{Crit}(\mathcal{L})$ with index $(q, \mathcal{L}) = k - 1$:

Case 1: Both are attractors of the slow equation.

- $p$ is an attractor of the slow equation.
- $q$ is an attractor of the slow equation.

Diagram:
- Blue arrows represent generic fast connections.
- Green arrows represent nongeneric fast connections.
- Red arrows represent slow orbits.

$k$ $k-1$ $k$ $k-1$ $k-1$

$p$ $q$

$k-1$ $k-2$ $k-1$
Theorem. Let

\[ p \in \text{Crit}(\mathcal{L}) \text{ with index } (p, \mathcal{L}) = k, \]
\[ q \in \text{Crit}(\mathcal{L}) \text{ with index } (q, \mathcal{L}) = k - 1 \]

The slow-fast orbits from \( p \) to \( q \) are in one-to-one correspondence with the trajectories from \( p \) to \( q \) for small \( \lambda \).

So the slow-fast orbits can be used to define a chain complex \( C^0 \) isomorphic to \( C^\lambda \), the chain complex for small \( \lambda \).

Proof. Geometric singular perturbation theory plus control afforded by the energy, with one little point.
Consider $\dot{y} = g(y, \varepsilon)$, $\varepsilon \geq 0$. Situation:

(1) $M_\varepsilon$ = cross section of tracked manifold.

(2) $N_\varepsilon$ = normally hyperbolic invariant manifold.

(3) $M_\varepsilon$ is transverse to $W^s(N_\varepsilon)$.

(4) $M_\varepsilon \cap W^s(N_\varepsilon)$ projects diffeomorphically along the stable fibration to $P_\varepsilon \subset N_\varepsilon$.

(5) $g$ is not parallel to $P$, at least at order $\varepsilon$.

(6) For $\varepsilon > 0$, in time of order $1/\varepsilon$, $P_\varepsilon$ becomes $P_\varepsilon^*$ with one higher dimension.

(7) $P_\varepsilon^*$ is a smooth perturbation of a manifold $P_0^*$.

**General Exchange Lemma.** Part of $M_\varepsilon^*$, $\varepsilon > 0$, is a smooth perturbation of $W^u(P_0^*)$. 
Replace (7) with:

\[ (7') \quad P_\epsilon^* \text{ is } C^r \text{ close to a manifold } P_{0*}^*. \]

**General Exchange Lemma v.2.** \( M_\epsilon^*, \epsilon > 0, \) is \( C^{r-1} \) close to \( W^u(P_{0*}) \).
Limit as $\lambda \to \infty$

Lagrange function:

$$\mathcal{L}(x, \eta) = f(x) + \eta \gamma(x).$$

Negative gradient flow with rescaled metric:

$$\dot{x} = - \left( \nabla f(x) + \eta \nabla \gamma(x) \right),$$
$$\dot{\eta} = - \lambda^2 \gamma(x).$$

Change of variables appropriate for large $\eta$:

$$\lambda = \frac{1}{\varepsilon}, \quad \eta = \frac{\rho}{\varepsilon}, \quad t = \varepsilon \tau.$$  

System becomes

$$\frac{dx}{d\tau} = - \left( \varepsilon \nabla f(x) + \rho \nabla \gamma(x) \right),$$
$$\frac{d\rho}{d\tau} = - \gamma(x).$$

Not a slow-fast system, but set $\varepsilon = 0$ (i.e. $\lambda = \infty$):
\[
\frac{dx}{d\tau} = -\rho \nabla \gamma(x)
\]
\[
\frac{d\rho}{d\tau} = -\gamma(x).
\]

Set of equilibria for $\varepsilon = 0$: $N_0 = \{(x, \rho) : \gamma(x) = 0 \text{ and } \rho = 0\}$.

$N_0$ is a compact codimension-two submanifold of $M \times \mathbb{R}$.

For $\varepsilon = 0$, $N_0$ is normally hyperbolic:

- Eigenvalues are 0 with multiplicity 2 and $\pm \| \nabla \gamma(x) \|_{g(x)}$. 

\[\varepsilon\]
\[
\begin{align*}
\frac{dx}{d\tau} &= - (\varepsilon \nabla f(x) + \rho \nabla \gamma(x)), \\
\frac{d\rho}{d\tau} &= -\gamma(x).
\end{align*}
\]

For small \(\varepsilon > 0\), there is normally hyperbolic invariant manifold \(N_\varepsilon\) near \(N_0\).

Locally chose coordinates on \(M\) with \(\gamma = x_n\).

Let \(y = (x_1, \ldots, x_{n-1})\), so \(x = (y, x_n)\).

Locally \(N_\varepsilon\) is parameterized by \(y\).

The system restricted to \(N_\varepsilon\), after division by \(\varepsilon\), is

\[
\dot{y} = -\nabla_y f(y, 0) + O(\varepsilon),
\]

where \(\nabla_y f(y, x_n)\) denotes the first \(n - 1\) components of \(\nabla f(y, x_n)\).

This is a perturbation of the negative gradient flow of \((f, g)\big|_{\gamma^{-1}(0)}\).

Assume: \((f, g)\big|_{\gamma^{-1}(0)}\) is Morse-Smale. Then its negative gradient flow is structurally stable.

Within \(N_\varepsilon\) we have the same equilibria and connections. The equilibria have one higher index.
\[
\frac{dx}{d\tau} = -(\varepsilon \nabla f(x) + \rho \nabla \gamma(x)),
\]
\[
\frac{d\rho}{d\tau} = -\gamma(x).
\]

Are there other connections?

- They are the only connections that stay in a neighborhood of \(N_\varepsilon\).
- \(E_\varepsilon(x, \rho) = \varepsilon f(x) + \rho \gamma(x)\) decreases along solutions.
- Since \(\gamma = \mathcal{O}(\varepsilon)\) on \(N_\varepsilon\), the energy difference between two equilibria is \(\mathcal{O}(\varepsilon)\).
- However, if a solution leaves a neighborhood of \(N_\varepsilon\), its energy drops by \(\mathcal{O}(1)\).

Let \(V = \{ (x, \rho) : |\gamma(x)| < \alpha, |\rho| < \alpha, |\rho \gamma(x)| < \frac{\alpha^2}{4} \}\).

Make a small \(\varepsilon\)-dependent alteration in \(V\): replace the portions of \(\partial V\) on which \(\gamma = \pm \alpha\) or \(\gamma = \pm \alpha\) by nearby invariant surfaces, so solutions can’t cross them.
Theorem. For $\lambda$ large, the Morse-Smale-Whitten complex of $(\mathcal{L}, g \oplus \lambda^{-2}e)$ equals that of $(f, g)|_{\gamma^{-1}(0)}$ with grading shifted by one.
The slow-fast flow and Morse theory

How does the Morse complex of $\gamma^{-1}(c)$ change as $c$ varies?

Replace $\gamma(x)$ by $\gamma(x) - c$. Then replace $L(x, \eta)$ by

$$L_c(x, \eta) = f(x) + \eta(\gamma(x) - c)$$

Rescaled ODE for the negative gradient flow of $L_c$

$$\dot{x} = - (\nabla f(x) + \eta \nabla \gamma(x)),
\dot{\eta} = -\lambda^2 (\gamma(x) - c).$$

- $S_L = S_{L_c}$.
- The fast flow does not change.
- The slow flow changes as the intersection of $S_L$ and $\gamma^{-1}(c) \times \mathbb{R}$ changes.

This should make it “easy” to check how slow-fast orbits appear and disappear as $c$ varies.
If no critical value of $\gamma$ is crossed, the homology of the chain complex should not change.

\[ I(x, f_\eta) = k - 1 \]

\[ I((x, \eta), L) = k \]

generic fast connection
nongeneric fast connection
If a critical value of $\gamma$ is crossed, the homology of the chain complex changes.

\[
\dot{x} = - (\nabla f(x) + \eta \nabla \gamma(x)), \\
\dot{\eta} = -\lambda^2 (\gamma(x) - c).
\]

The slow flow changes at $\eta = \pm \infty$. 