Due Thursday, March 2

(1) Problem 3.2-3 in Lewis and Syrmos.

(2) Find a curve that is an extremal for the functional

\[ J(x) = \int_0^{t_f} \sqrt{1 + \dot{x}^2(t)} \, dt \]

such that \( x(0) = 5 \) and \( x(t_f) \) lies on the circle \( x^2(t) + (t - 5)^2 - 4 = 0 \). Verify your solution geometrically.

(3) Find an extremal for the functional

\[ J(x) = \int_0^{\pi/2} \left\{ \dot{x}_1^2(t) + \dot{x}_2^2(t) + 2x_1(t)x_2(t) \right\} \, dt \]

where

\[ x_1(0) = 0 \quad x_1(\pi/2) = \text{free} \]
\[ x_2(0) = 0 \quad x_2(\pi/2) = 1 \]

(4) Consider a surface of revolution created by revolving a curve \( y(x) \) about the \( x \)-axis. The curve is required to pass through the fixed end points \( (x_1, y_1) \) and \( (x_2, y_2) \). Use variational principles to determine the curve \( y(x) \) which minimizes the resulting surface area. Minimum surface problems of this type are often called “soap film” problems.

(5) **Hamilton’s Principle in Mechanics**: The motion of a conservative system from time \( t_0 \) to \( t_f \) is such that the integral

\[ J(u) = \int_{t_0}^{t_f} L(u(t), x(t)) \, dt \]

is minimized. Here

- \( x \) is the generalized coordinate vector (state variable)
- \( u = \dot{x} \) is the generalized velocity vector
- \( L = T(u, x) - U(x) \) is the Lagrangian of the system
- \( T \) is the kinetic energy for the system
- \( U \) is the potential energy for the system.

(a) In mechanics, the Hamiltonian is defined as

\[ H = -L + p^T u \]
so that it represents the total energy (we use the definition $H = L + p^T u$ for optimal control formulation). Determine the optimality conditions and show that they yield the Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0.$$ 

(b) Show that $H = T + U$ is constant on optimal trajectories (hence the sum of the kinetic and potential energies is constant during motion). Recall that $T = \frac{1}{2} m \dot{x}^2$ so that $\frac{\partial T}{\partial u} u = 2T$.

(c) Consider a pendulum of length $\ell$ and mass $m$. The generalized coordinate is taken to be the angle $\theta$ of the pendulum from vertical. If we let $x = \ell \theta$ and $u = \ell \dot{\theta}$, the kinetic and potential energies are given by

$$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \ell^2 \dot{\theta}^2$$
$$U = mg\ell(1 - \cos \theta).$$

Use the Euler-Lagrange equations to compute the equation of motion for the pendulum. Compare to the familiar pendulum equation obtained using Newton’s laws

$$\frac{d}{dt}(m \dot{x}) + \frac{\partial U}{\partial x} = 0.$$