MA 574 – PROJECT 3

Due: Wednesday, April 13

(1) Consider a rod of length $L$ with a fixed end at $x = 0$ as depicted in Figure 1. Let $\rho, Y, c$ respectively denote the density, Young’s modulus and internal damping coefficient and let $u(t, x)$ denote the longitudinal displacements. The rod decreases from a thickness $2h$ at $x = 0$ to a thickness of $h$ at the free end and has uniform width $b$. Moreover, the rod has a mass $m_L$ at $x = L$ along with a restraining spring with stiffness $k_L$ that exerts a restoring force proportional to displacements $u(t, L)$.

(a) Balance forces to obtain a strong formulation of the model. Be sure to specify boundary conditions.

(b) Integrate by parts to obtain a corresponding weak formulation. Specify the space of test functions.

(c) Determine an appropriate basis and finite element solution to (b). Determine the resulting matrix vector system in second-order and first-order form. You should specify the components in your mass and stiffness matrices but you do not need to numerically evaluate the integrals.

(d) Consider the undamped rod, $c = 0$, with $m_L = k_L = 0$. Determine the kinetic and potential energies and use Hamiltonian (energy) principles to derive a weak formulation of the model. How does it compare with the model that you derived in (b)?

\[ 0 \leq x \leq L \]

Figure 1: Geometry of the rod for Problem 1.

(2) Consider the weak formulation

\[
\int_0^\ell \rho A \frac{\partial^2 u}{\partial t^2} \phi dt + \int_0^\ell \left[ Y A \frac{\partial u}{\partial x} + C A \frac{\partial^2 u}{\partial x \partial t} \right] \frac{d\phi}{dx} = - \left[ k_\ell u(t, \ell) + c_\ell \frac{\partial u}{\partial t}(t, \ell) + m_\ell \frac{\partial^2 u}{\partial t^2}(t, \ell) \right] \phi(\ell),
\]

which holds for all $(\phi, \phi(\ell)) \in V$. Using the linear hat functions defined in class, this yields the first-order system

\[
\frac{dz}{dt} = Az(t)
\]

\[
z(0) = z_0.
\]

Using the values $\rho = 2.7 \text{ g/cm}^3$, $Y = 70 \text{ GPa}$, $C = 10 \text{ MPa}$, $A = 1 \text{ cm}^2$, $k_\ell = 10 \text{ MPa}$, $c_\ell = 10 \text{ Pa}$, $m_\ell = 1 \text{ g}$, and $\ell = 10 \text{ cm}$, construct the matrix $A$ and compute its eigenvalues. Do all of them have negative real part? Now construct the matrix $A$ when the right hand side is positive and discuss the sign of the eigenvalues.
(3) Consider a thin beam of length $L$ with a fixed end at $x = 0$ as depicted in Figure 2. Let $\hat{\rho}, Y, C$ and $\gamma$ respectively denote the density, Young’s modulus, internal damping coefficient and air damping coefficient for the beam and let $w(t, x)$ denote transverse displacements. The beam decreases from thickness $2h$ at $x = 0$ to a thickness of $h$ at the free end and has uniform width $b$. The $z$-axis is taken to be the vertical axis and the bottom of the beam has the coordinate $z = -\frac{h}{2}$. Finally, the beam has a mass $m_L$ at $x = L$ along with a restraining spring with stiffness $k_L$ that exerts a restoring force proportional to displacements $w(t, L)$.

(a) Define the neutral line and compute its position $n(x)$. Draw the neutral line on the figure of the beam.

(b) Determine relations for the linear density $\rho(x)$ and stiffness $YI(x)$.

(c) Balance moments and forces to obtain a strong formulation of the beam model.

(d) Specify appropriate boundary conditions.

(e) Integrate by parts to obtain a corresponding weak formulation. Be sure to specify the space of test functions.

(f) Consider the undamped beam ($\gamma = C = 0$) with $m_L = k_L = 0$. Use Hamiltonian (energy) principles to derive a weak formulation of the model. How does it compare with the model that you derived in (e)?

Figure 2: Thin vibrating beam.