The Golden Mean and Fibonacci Numbers

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1 The golden mean

A version of an old mathematical joke goes as follows:

Theorem All numbers are interesting.

Proof. Let \( U \) be the set of noninteresting nonnegative numbers. If \( U \) is nonempty, it is bounded below, and therefore has a greatest lower bound \( a \). If \( a \) is in \( U \) then there is a smallest noninteresting number; but this is interesting. If \( a \) is not in \( U \) then there is no smallest noninteresting number; but this is also interesting. Therefore all positive numbers are interesting. Similar arguments may be given for negative numbers. Thus all numbers are interesting. \( \square \)

Although all numbers are interesting, some are more interesting than others. A case in point is the number

\[
\frac{1 + \sqrt{5}}{2},
\]

known as the golden mean, the golden ratio or the golden section. Why is this number given such a exalted name? Indeed this number, known from the time of the ancient Greeks, does pop up in a number of geometrical constructions and has a number of interesting mathematical properties. However, these considerations alone do not seem to account for the fancy name. Over the years, there have been many attempts at connecting the golden mean to art, architecture, biology, botany and other fields. This has resulted in a cult of enthusiasts that have attributed almost mystical properties to this number and resulted in the names of such as the golden mean and divine proportion. We shall discuss some of these connections in section 15. However our focus will be on the mathematics associated with the golden mean.
We begin with a strangely worded proposition in Euclid: “To divide a line segment in extreme and mean ratio.” By this is meant the division of a line segment so that the whole is to the larger part as the larger part is to the smaller part.

| a | b |

If we denote the larger part by \(a\) and the smaller part by \(b\), this requirement is

\[
\frac{a+b}{a} = \frac{a}{b}, \quad \text{or} \quad a^2 = a(a+b).
\]

The second equation above shows that the larger part is the geometric mean of the smaller part and the whole. Putting \(x = a/b\), we find

\[
x^2 = x + 1.
\] (1)

The golden mean, often symbolized by \(\phi\), is the positive root of this equation:

\[
\phi = \frac{1 + \sqrt{5}}{2} = 1.61803989\ldots
\] (2)

See Section 15 for an explanation of why the symbol \(\phi\) is used for this number. It follows immediately from (1) that the golden mean has the property that adding one produces its square and subtracting one produces its reciprocal:

\[
\phi^2 = \phi + 1, \quad \phi^{-1} = \phi - 1.
\]

The other root of equation (1) is a negative number

\[
\psi = 1 - \phi = -\frac{1}{\phi} = \frac{1 - \sqrt{5}}{2} = -0.61803989\ldots
\]

2 The golden mean in geometry

It is an easy matter to construct the golden mean with ruler and compass. Start with a unit square \(ABCD\) as shown in figure 1. Connect the midpoint \(E\) of side \(AB\) with the vertex \(C\). The line \(EC\) has length \(\sqrt{5}/2\). Using the point \(E\) as center and the line \(EC\) as radius draw and arc \(CF\) meeting the line \(AB\) extended at the point \(F\). The line \(AF\) has length \((1 + \sqrt{5})/2 = \phi\), the golden mean. A rectangle such as \(AFGD\), whose sides are in the ratio 1 : \(\phi\), is called golden rectangle.
Another proposition in Euclid calls for the construction of an isosceles triangle whose base angle is twice the vertex angle, that is, an isosceles triangle with a base of $72^\circ$ and a vertex angle of $36^\circ$. We shall see that this involves the construction of the golden mean. In Figure 2 assume that the base, $AC$, of the isosceles triangle $ABC$ has length 1 and the side $BC$ has length $x$. If the angle at $C$ is bisected we obtain a triangle $ADC$ which is similar to $ABC$. It follows at once that

$$x/1 = 1/(x-1) \quad \text{or} \quad x^2 = x + 1. \quad (3)$$

Thus the ratio of the side to the base of the isosceles triangle $ABC$ is $\phi$, the golden mean; we call this the *thin* golden triangle. In the isosceles triangle $CDB$ the ratio of the base to a side is the golden mean; we call this the *fat* golden triangle. The thin golden triangle has a base angle of $72^\circ$, whereas the fat golden triangle has a vertex angle of $36^\circ$ as shown in Figure 3.
It is now a simple matter to see that the regular pentagon and the regular
decagon may be constructed by ruler and compass. The regular pentagon may be
disected into two fat and one thin golden triangles as shown in Figure 4. A diag-
onal and a side of a regular pentagon are in the golden ratio. The regular decagon
consists of ten thin golden rectangles as shown in Figure 5; the circumradius and
an edge of a regular decagon are in the golden mean.

Drawing all the diagonals of a regular pentagon produces a regular five-pointed
star inside the pentagon as shown in Figure 6. This figure is called a pentagram.
The pentagram is loaded with golden ratios. If the side of the outer pentagon is
1, then there are 5 line segments of length $\phi^{-2}$, 10 of length $\phi^{-1}$, 15 of length 1,
and 5 of length $\phi$. Thus there are 275 distinct occurrences of the golden ratio in the
pentagram.
Perhaps the most amazing occurrence of the golden mean in geometry is in connection with the regular icosahedron and regular dodecahedron. A regular icosahedron has 20 faces and 12 vertices. We may describe the vertices of the icosahedron as follows. We take three golden rectangles in the three coordinate planes as shown in Figure 7. If the golden rectangles have sides of length 2 and $2\phi$, these vertices are $(\pm 1, 0, \pm \phi), (0, \pm \phi, \pm 1)$ and $(\pm \phi, \pm 1, 0)$. Connecting each vertex with its closest neighbors produces the edges of the 20 equilateral triangular faces. Look at any vertex, say, the vertex $P$ in Figure 7. Emanating from the vertex $P$ are 5 equilateral triangles which form a prism whose base $QRSTU$ is a regular pentagon. One diagonal $TR$ is the longer edge of one of the golden rectangles, which has length $2\phi$, while the edge $PS$ is the shorter side of another golden rectangle and has length 2. Thus the diagonals and edges of the regular icosahedron are in the golden ratio.

A regular Dodecahedron has 12 regular pentagonal faces and 20 vertices as shown in Figure 8. The regular dodecahedron and the regular iscoahedron are dual polyhedra; that is, the centers of the faces of a regular icosahedron from the vertices of a regular dodecahedron and vice-versa. Thus the centers of the faces of a regular dodecahedron are the vertices of three mutually perpendicular golden rectangles referred to above.
Figure 7: Regular Icosahedron

Figure 8: Regular Dodecahedron
3 A golden spiral

A logarithmic spiral\(^1\) may be associated with a golden rectangle. Start with a golden rectangle \(ABCD\) as shown in Figure 9 where the length of \(AB\) is \(\phi\) and the length of \(AD\) is 1. Since \(\phi = 1 + \frac{1}{\phi}\), we may divide the golden rectangle \(ABCD\) into a square \(AFED\) and a smaller golden rectangle \(FBCE\) whose sides have been reduced by a factor of \(\frac{1}{\phi}\). This smaller golden rectangle may be similarly subdivided and the process continued indefinitely. If the successive subdivisions are oriented properly, we obtain a sequence of nested golden rectangles which have a single point \(O\) in common.

Let us now perform the subdivision using the construction shown in Figure 10. In the rectangle \(ABCD\) we draw the diagonal \(AC\) and construct the line \(CE\) perpendicular to \(AC\) intersecting the diagonal at the point \(O\). Now construct the perpendicular \(EF\). It is easily seen that the rectangle \(FBCE\) is similar to \(ABCD\). Furthermore, the point \(O\) has the same relative position in \(FBCE\) as it does in \(ABCD\). The next smallest rectangle may be obtained by by constructing a perpendicular at \(H\) to obtain the rectangle \(GHCF\). The process may be repeated ad infinitum forming a infinite sequence of nested golden rectangles which have the unique point \(O\) in common to all the nested golden rectangles.

Another way to obtain the successive golden rectangles is as follows. The

\(^1\)A logarithmic spiral is a curve which makes a constant angle, \(\alpha\), with the radius vector. In polar coordinates it has an equation of the form \(r = ae^{b\theta}\) where \(b = \cot \alpha\). As the angle \(\theta\) increases in an arithmetic progression, the radius increases in a geometric progression.
Rectangle \( FBCE \) may be obtained from \( ABCD \) by first rotating \( ABCD \) about the point \( O \) in the positive direction and shrink each point towards \( O \) by a factor of \( \phi^{-1} \) as shown in Figure 11. Note that the vertices \( A, B, C, D \) are transformed to \( B, C, E, F \) respectively. This process may be repeated to obtain all of the nested rectangles. The positions of the point \( A \) and it’s successive transforms \( C, E, F, \ldots \) may be described using polar coordinates with \( O \) as the pole and \( OA \) as the initial line. If we denote the length of \( OA \) by \( c \), these coordinates are

\[
A = (c, 0), \quad B = (c\phi^{-1}, \pi/2), \quad C = (c\phi^{-2}, 2\pi/2), \quad E = (c\phi^{-3}, 3\pi/2), \ldots
\]

In general the transforms of \( A \) are

\[
r = c\phi^{-t}, \quad \theta = t\pi/2, \quad t = 0, 1, 2, 3, \ldots
\]

Now letting \( t \) be any real number, all these points lie on the logarithmic spiral

\[
r = c\phi^{-2\theta/\pi}.
\]

Logarithmic spirals, or at least approximate ones, seem to be prevalent in nature—the shell of the chambered nautilus, elephant tusks, rams horns, and so on. This is sometimes given as evidence of the intimate connection between the golden mean and natural phenomena. However, as we shall see, one can associate a logarithmic spiral with any rectangle in a manner similar to what is done for the golden rectangle. Thus the golden rectangle is not really special in this regard.
Start with any rectangle of sides $a$ and $b$ with $a > b$. Using the same construction as shown in Figure 13 we may construct a smaller similar rectangle whose sides are reduced by the factor $\lambda^{-1}$ where $\lambda = a/b$. We may therefore construct a sequence of similar rectangles converging to the point $O$ and a logarithmic spiral whose equation is

$$r = c\lambda^{-2\theta/\pi}.$$ 

The only thing special about the spiral associated with the golden rectangle is that the factor $\lambda$ is equal to the golden mean.

## 4 More Spirals

### 4.1 Spiral based on a Thin golden triangle

We may also associate a logarithmic spiral with a thin golden isosceles triangle in a manner similar to the spiral for a golden rectangle shown in section 3. We begin with a thin golden isosceles triangle $ABC$ as shown in Figure 14. As we saw in Section 2, bisecting the angle at vertex $A$ produces a similar isosceles triangle $BCD$ whose sides have been reduced by a factor of $\phi^{-1}$. Bisecting the angle at $D$ produces another similar isosceles triangle $DEF$. We may continue this process to obtain a nested sequence of triangles which have a point $O$ in common.

If we denote the midpoint of $AB$ by $M$ and the midpoint of $BC$ by $N$, the point $O$ is the intersection of the lines $BM$ and $DN$ as shown in Figure 15. Furthermore
the point $O$ has the same relative position in the triangle $BCD$ as it has in the triangle $ABC$. An alternative way of obtaining the triangle $BCD$ is to rotate the triangle $ABC$ about the point $O$ by an angle of $3\pi/5$ and then shrink it towards $O$ by a factor of $\phi^{-1}$. We may continue this process of rotation and dilation to obtain the other nested triangles shown in Figure 15.

A logarithmic spiral may be drawn through the vertices $A, B, C, D, E, F, G, H, \ldots$ as follows. The positions of the point $A$ and its successive transforms $B, C, D, \ldots$ may be described using polar coordinates with $O$ as the pole and $OA$ as the initial line. If we denote the length of $OA$ by $c$, these coordinates are

$$A = (c, 0), B = (c\phi^{-1}, \frac{3\pi}{5}), C = (c\phi^{-2}, 2\frac{3\pi}{5}), D = (c\phi^{3}, 3\frac{3\pi}{5}), \ldots$$

(4)
In general the transforms of $A$ are

$$r = c\phi^{-t}, \quad \theta = t \cdot \frac{3\pi}{5}, \quad t = 0, 1, 2, 3, \ldots$$

Now letting $t$ be any real number, all these points lie on the logarithmic spiral

$$r = c\phi^{-5\theta/3\pi}$$

as shown in Figure 16. Note that this spiral passes through each vertex of each of the nested triangles. Thus it seems a better fit that the spiral associated with the golden rectangle which passes through 3 of the 4 vertices in each of the nested rectangles.
Figure 16: Thin Triangle Spiral
4.2 A Spiral based on the Kepler Triangle

An interesting spiral, called the Theodorus Spiral, or square root spiral, dates back to the ancient Greek times. It is formed starting with a isosceles right triangle with sides of 1 and then constructing additional right triangles with the hypotenuse of the first as one side and unity as the other side as shown in Figure 17. The new hypotenuse has a length of $\sqrt{2}$. This process is continued producing hypotenuses of length $\sqrt{3}, \sqrt{4}, \ldots$.

![Figure 17: Theodorus Spiral](image)

The properties of the Theodorus spiral, and the development of a smooth curve from it, are discussed in [3]. This spiral is not connected with the golden mean. However it is the inspiration for Oster [8] to develop a spiral that is connected to the golden mean.

The Oster spiral is based on a Kepler triangle, that is, a right triangle whose sides and hypotenuse form a geometric progression. If we make one side of length 1 and the other side of length $a$, the hypotenuse must be of length $a^2$ where $a$ is the geometric ratio. The Pythagorean theorem gives us $a^4 = a^2 + 1$. This implies that $a^2 = \phi$ and $a = \sqrt{\phi}$. Therefore any Kepler triangle must be similar to one of edges 1, $\sqrt{\phi}$, $\phi$ as shown in Figure 18. The smaller angle $\alpha$ of a Kepler Triangle is $\alpha = \arctan(1/\sqrt{\phi}) = 0.666239432\ldots$ radians. An interesting fact about the angle $\alpha$
is that \( \tan \alpha = \cos \alpha = \frac{1}{\sqrt{\phi}} \).

![Figure 18: A Kepler Triangle](image)

To construct a rectilinear spiral based on Kepler triangle we start with a Kepler triangle of sides \( \sqrt{\phi} \) and 1 in the first quadrant as shown in Figure 19. We then construct an additional similar Kepler triangle using the hypotenuse of the first triangle as the longer leg of the second triangle as shown in the figure. The process is then repeated. Each new triangle has sides that \( \sqrt{\phi} \) times the previous triangle. The triangles are all similar and subtend a base angle of \( \alpha \) at the origin.

The polar coordinates of the vertices of the Oster rectilinear spiral are

\[
(\phi^{1/2}, 0), (\phi^{3/2}, \alpha), (\phi^{3/2}, 2\alpha), \ldots,
\]

or

\[
(\phi^{(t+1)/2}, t\alpha), t = 0, 1, 2, \ldots.
\]

Now letting \( t \) be any real number we obtain another logarithmic spiral whose equation in polar coordinates is

\[
r = \phi^{1/2} \phi^{t/2\alpha},
\]

as shown in Figure 20.
Figure 19: Rectilinear Spiral based on a Kepler Triangle

Figure 20: Logarithmic Spiral based on a Kepler Triangle
5 Expressions for the Golden Mean

5.1 Decimal Expansion of the Golden Mean

Since the golden mean is equal to \((1 + \sqrt{5})/2\), it is clear that it is irrational. This means that \(\phi\) has an infinite non-repeating decimal expansion. To thirty places we have

\[
\phi = 1.61803398874989484820458683436\ldots
\]

An interesting direct proof of the irrationality of \(\phi\) may be given based on the fact the

\[
\phi^{-1} = \phi - 1.
\]

It follows easily by induction that

\[
\phi^{-n} = a_n\phi + b_n, \quad n = 1, 2, \ldots
\]  

(5)

where \(a_n\) and \(b_n\) are integers. Now assume that \(\phi\) is rational, that is, \(\phi = a/b\) where \(a\) and \(b\) are positive integers. From equation (5) we find

\[
b\phi^{-n} = a_na + b_nb, \quad n = 1, 2, \ldots
\]  

(6)

Consider the numbers \(c_n = b\phi^{-n}\). From equation (6), we see that the \(c_n\) are integers for each \(n\). Furthermore, since \(\phi > 0\), the \(c_n\) are positive integers for each \(n\) and, since \(\phi > 1\), the \(c_n\) strictly decrease as \(n\) increases. Therefore we have constructed an infinite decreasing sequence of positive integers, which is clearly impossible.

5.2 The Golden Mean in terms of Nested Square Roots

Although there is no simple formulae for the \(n\)-th digit in the decimal expansion of \(\phi\), there are other ways to express the numerical value of \(\phi\) that possess certain regularities. One expression is in terms of nested square roots.

We start with the fact that the golden mean is the positive root of

\[
x = \sqrt{1+x}.
\]

The form of this equation suggests a solution by iteration. we define

\[
x = 1, \quad x_{n+1} = \sqrt{1+x_n}, \quad n = 1, 2, 3, \ldots
\]
This yields

\[ x_2 = \sqrt{1 + \sqrt{1}} \]
\[ x_3 = \sqrt{1 + \sqrt{1 + \sqrt{1}}} \]

Thus, if the sequence converges, we have the golden mean expressed as an infinite nest of square roots of 1:

\[ \phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}} \]

The iteration may be illustrated geometrically by drawing the curves, \( y = x \) and \( y = \sqrt{1 + x} \) which intersect at the golden mean as shown in Figure 1. Start at \( x_1 = 1 \) and move up to the point \( P \) on curve \( y = \sqrt{1 + x} \). The ordinate of \( P \) is \( \sqrt{1 + x_1} = x_2 \), but this is the same as the abscissa of the point \( Q \) which is the intersection of the horizontal line through \( P \) and the curve \( y = x \). The successive iterates may be obtained in a similar manner, as shown in Figure 21.

![Figure 21: The sequence \( x_{n+1} = \sqrt{1 + x_n} \)](image)

It is clear from Figure that the successive iterates will converge to the golden mean. Note, also, that any positive number could be assigned for \( x_1 \) and the same limit would be achieved.
An analytical proof may also be given. We estimate the difference between $\phi$ and $x_{n+1}$:

$$|\phi - x_{n+1}| = \frac{|\phi^2 - x_{n+1}^2|}{|\phi + x_{n+1}|} = \frac{|\phi^2 - 1 - x_n|}{|\phi + x_{n+1}|} = \frac{|\phi - x_n|}{|\phi + x_{n+1}|} < \frac{|\phi - x_n|}{\phi}$$

using this fact repeatedly we find

$$|\phi - x_{n+1}| < \phi^{-1}|\phi - x_n| < \phi^{-2}|\phi - x_{n-1} < \cdots < \phi^{-n}|\phi - x_1|.$$ 

Since $\phi > 1$, the sequence $x_n$ converges to $\phi$.

### 5.3 A Continued Fraction for the Golden Mean

$\phi$ may also be described as the positive root of the equation

$$x = 1 + \frac{1}{x}.$$ 

(7)

Solving this by iteration we have

$$x_1 = 1, \quad x_{n+1} = 1 + \frac{1}{x_n}, \quad n = 1, 2, 3, \ldots$$

Figure 22: The sequence $x_{n+1} = 1 + 1/x$
This produces the sequence

\[ x_2 = 1 + \frac{1}{1} \]
\[ x_3 = 1 + \frac{1}{1 + \frac{1}{1} } \]

and so on. This is illustrated in Figure (22). This yields an expression for the golden mean as an infinite continued fraction consisting of only ones:

\[ \phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ddots} } } \]  

(8)

An analytic proof that \( x_n \) converges to \( \phi \) may be given as follows. Note that

\[ |\phi - x_{n+1}| = |\phi - \left(1 + \frac{1}{x_n}\right)| = \left| \frac{x_n(\phi - 1) - 1}{x_n} \right| = \left| \frac{x_n - \phi}{\phi x_n} \right| \leq \phi^{-1} |x_n - \phi| \]

From this it follows that

\[ |\phi - x_{n+1}| \leq \phi^{-1} |x_n - \phi| \leq \phi^{-1} |\phi - x_{n-1}| \leq \cdots \leq \phi^{-n} |\phi - x_1| \]

Since \( \phi > 1 \) we see that \( x_n \) converges to \( \phi \).

5.4 Another Expression for \( \phi \)

Let us start with the following formula for the golden mean.

\[ x = \sqrt{1+x} = \sqrt{2 + \frac{1}{x}}. \]

Proceeding as in the preceding examples we find a combined nested square root and continued fraction for \( \phi \):

\[ \phi = \sqrt{2 + \frac{1}{\sqrt{2 + \frac{1}{\sqrt{2 + \ddots} } } } } \]
6 A Golden Network

Let us now look at the infinite ladder network constructed by alternately connecting unit (1 ohm) resistances in series and parallel as shown in Figure 23. What is the resistance, \( R \), of this network? One way to solve this problem is to note that adding one more stage at the beginning will not change the value of \( R \) as shown in Figure 24. Now using the usual rules for series and parallel resistances we find

\[
R = 1 + \frac{1}{\frac{1}{1/R}}.
\]  

(9)

This simplifies to

\[
R^2 = R + 1,
\]  

(10)

so that \( R = \phi \), the golden mean.

Another way of finding the resistance of this infinite network is to find the resistance \( R_n \) of a network with \( n \) stages and then let \( n \) approach infinity. We add an additional stage, as shown in Figure 25. Using the laws for series and parallel connections, we find

\[
R_1 = 2, \quad R_{n+1} = 1 + \frac{R_n}{1+R_n} = 1 + \frac{1}{1 + \frac{1}{R_n}}.
\]  

(11)
As \( n \) approaches infinity, we see that \( R_n \) approaches the infinite continued fraction shown in Equation (8), that is, the golden mean.

\[
\begin{array}{c}
R_{n+1} \quad R_n \\
\hline
\text{n stages}
\end{array}
\]

Figure 25: \( R_{n+1} \) in terms of \( R_n \)

7 The Rabbit Problem and Fibonacci Numbers

One of the few mathematicians of stature in the Middle Ages was the Italian, Leonardo of Pisa, who was nicknamed Fibonacci. In 1202 Fibonacci wrote an important work entitled ‘Liber Abacci’ which contained much of the arithmetic and algebraic knowledge of the time. This book exerted a considerable influence on the development of mathematics in Western Europe; in particular it helped to advance the use of of Arabic numerals and positional notation. We are concerned here with one innocent little problem which appeared in “Liber Abacci”:

On the first day of a month we are given a new born\(^2\) pair of rabbits. It is assumed that no rabbits die, that they begin to bear young when they are two months old, and that they produce one pair of rabbits on the first day of each month thereafter. How many pairs of rabbits will there be exactly one year later?

Let \( F_n \) be the number of pairs of rabbits at the end of the \( n \)-th month. The number of pairs of rabbits at the end of the \( (n + 2) \)-th month, \( F_{n+2} \), must satisfy the following equation

\[
F_{n+2} = F_{n+1} + \text{births during the } (n+2)\text{-th month}
\]

Each pair of rabbits that is at least 2 months old will produce a pair during month \( n + 2 \); there are precisely \( F_n \) such pairs. Therefore we have the recurrence relation,

\(^2\)Fibonacci’s original problem started with a one month old pair of rabbits.
or difference equation:

\[ F_{n+2} = F_{n+1} + F_n, \quad n = 1, 2, 3, \ldots \]  \hspace{1cm} (12)

with the initial conditions

\[ F_1 = 1, \quad F_2 = 1. \]  \hspace{1cm} (13)

The difference equation (12), together with the initial conditions (13), uniquely define a sequence of numbers \( F_n \) called Fibonacci numbers. It is easy to write down the first few Fibonacci numbers since each number is the sum of the previous two numbers. The first twenty Fibonacci numbers are

\[
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1507, 1584, 4181, 6765
\]  \hspace{1cm} (14)

The twelfth number of the sequence is \( F_{12} = 144 \) which is the number of pairs of rabbits at the end of the twelfth month. However, exactly one year after the start would be the start of the 13-th month, and on that date there would be 233 pairs of rabbits.

8 Binet’s formula for \( F_n \)

It is natural to look for a formula for the \( n \)-th Fibonacci number. We begin by seeing how fast the Fibonacci numbers grow. Let

\[ x_n = F_{n+1}/F_n. \]  \hspace{1cm} (15)

Using (14), we find the first few terms of the sequence \( x_n \) are

\[
1, 2, 1.5, 1.6666..., 1.6, 1.625, 1.6154, 1.619, 1.6176, 1.6182.
\]  \hspace{1cm} (16)

It appears that \( x_n \) approaches the golden mean. This is not difficult to prove. From equations (15) and (12 we find

\[ x_{n+1} = 1 + 1/x_n, \quad n = 1, 2, 3, \ldots, \quad x_1 = 1. \]  \hspace{1cm} (17)

This is precisely the difference equation discussed in section 5.3, where it was shown that \( x_n = F_{n+1}/F_n \to \phi \) as \( n \to \infty \). Thus, for large \( n \), the Fibonacci numbers grow like powers of the golden mean. Let us take a closer look at these powers:

\[
1, \phi, \phi^2, \phi^3, \ldots, \phi^n, \ldots
\]  \hspace{1cm} (18)
Using the fact that $\phi^2 = \phi + 1$, we find
\[ \phi^{n+2} = \phi^n \phi^2 = \phi^n (\phi + 1) = \phi^{n+1} + \phi^n. \]  
(19)

We see that $\phi_n$ satisfies the same difference equation as the Fibonacci numbers. By successively using the fact that $\phi^2 = \phi + 1$ we find
\[ \phi^3 = \phi^2 \phi = 2\phi + 1, \]
\[ \phi^4 = \phi^3 \phi = (2\phi + 1)\phi = 3\phi + 2, \]
\[ \phi^5 = 5\phi + 3, \]
\[ \phi^6 = 8\phi + 5, \]
and so on. This suggests that
\[ \phi^n = F_n \phi + F_{n-1}, \quad n = 2, 3, \ldots, \]
(20)
a fact that may easily be proved by induction. Since only the fact that $\phi^2 = \phi + 1$ was used to establish equation (20), a similar equation must also hold for the other root of $x^2 = x + 1$, namely $\psi = 1 - \phi = (1 - \sqrt{5})/2 = -0.61803989\ldots$, that is
\[ \psi^n = F_n \psi + F_{n-1} \quad n = 2, 3, \ldots, \]
(21)
Solving the two equations (20) and (21) for $F_n$ we find
\[ F_n = \frac{\phi^n - \psi^n}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right). \]
(22)
This is usually called Binet’s formula. It is interesting to note that the right hand side of Binet’s formula must be an integer even though it involves fractions and square roots. Binet’s formula is mainly of theoretical interest; we will use it in the next section to derive several identities.

Direct use of the formula to calculate Fibonacci numbers is practical only for small values of $n$. Is there any alternative to calculating, say $F_{60}$, other than to use the recurrence formula over and over again? One way is to note that, from Binet’s formula, we may deduce:
\[ \left| F_n - \phi^n / \sqrt{5} \right| = |\psi^n / \sqrt{5}| < 1/2. \]  
(23)
This means that $F_n$ is the nearest whole number to $\phi^n / \sqrt{5}$. Using this result we may find $F_{60}$ on an ordinary scientific calculator. The computation yields $\phi^{60} / \sqrt{5} = 1548008755920.003$, thus $F_{60} = 1548008755920$. 

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9 A Few Fibonacci Identities

Before deriving these identities, it is useful to set \( F_0 = 0 \); the Fibonacci numbers may now be defined as

\[
F_0 = 0, \ F_1 = 1, \ F_{n+2} = F_{n+1} + F_n, \ \text{n} = 0, 1, 2, \ldots
\]

Binet’s formula also holds for \( n = 0 \) so that

\[
F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}, \ \text{n} = 0, 1, 2, \ldots
\]

We also have the useful identities:

\[
\phi^2 = \phi + 1, \quad \phi^{-1} = \phi - 1, \quad \psi^2 = \psi + 1, \quad \psi^{-1} = \psi - 1, \quad (24)
\]

\[
\phi^n = F_n\phi + F_{n-1}, \quad \psi^n = F_n\psi + F_{n-1}, \quad (25)
\]

\[
\phi + \psi = 1, \quad \phi\psi = -1, \quad \phi - \psi = \sqrt{5}. \quad (26)
\]

Now let us put the above facts to work by finding the sum of the first \( n \) Fibonacci numbers.

\[
F_1 + F_2 + \cdots + F_n = \sum_{k=1}^{n} F_k = \sum_{k=0}^{n} F_k = \sum_{k=0}^{n} \frac{\phi^k - \psi^k}{\sqrt{5}} \quad (27)
\]

However

\[
\sum_{k=0}^{n} \phi^k = \frac{1 - \phi^{n+1}}{1 - \phi} = \phi^{n+2} - \phi, \quad \sum_{k=0}^{n} \psi^k = \frac{1 - \psi^{n+1}}{1 - \psi} = \psi^{n+2} - \psi. \quad (28)
\]

Thus

\[
F_1 + F_2 + \cdots + F_n = \sum_{k=1}^{n} F_k = \frac{\phi^{n+2} - \psi^{n+2} - (\phi - \psi)}{\sqrt{5}}
\]

\[
= F_{n+2} - 1. \quad (29)
\]

In similar way one may derive (rather than verify) the following identities.

\[
F_1 + F_3 + F_5 + F_{2n-1} = F_{2n},
\]

\[
F_1 - F_2 + F_3 + \cdots + (-1)^{n+1} F_n = (-1)^{n+1} F_{n-1} + 1,
\]

\[
F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1},
\]

\[
F_1 F_2 + F_2 F_3 + \cdots + F_{2n-1} F_{2n} = F_{2n}^2
\]

\[
F_n^2 + F_{n+1}^2 = F_{2n+1}
\]
These identities may also be proved by mathematical induction. For example we give an inductive proof of Equation (29), $\sum_1^n F_k = F_{n+2} + 1$. Note that for $n = 1$ we find that $F_1 = F_3 - 1 = 1$. Now assuming that the statement is true for $n$ we must show it is true for $n + 1$. Adding $F_{n+1}$ to both sides yields
\[
\sum_1^n F_k + F_{n+1} = F_{n+2} + 1 + F_{n+1}.
\]
This may be written as
\[
\sum_1^{n+1} F_k = F_{n+3} + 1,
\]
which is exactly Equation (29) with $n$ replaced by $n + 1$.

10 A Matrix Version of the Fibonacci Numbers

The second order difference equation defining Fibonacci numbers, $F_{n+1} = F_{n+1} + F_n$, may be expressed as a system of first order difference equations. Consider the system of two equations
\[
F_{n+1} = F_n + F_{n+1},
F_n = F_n,
\]
for $n = 1, 2, 3, \ldots$. Putting this into matrix form yields
\[
\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = Q \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}, \quad n = 1, 2, 3, \ldots \quad (30)
\]
where $Q$ is the matrix
\[
Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}
\]
By successive substitution of $n = 1, 2, 3, \ldots$ we find
\[
\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = Q^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = Q^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad n = 1, 2, 3, \ldots \quad (31)
\]
If we did not already have the Binet formula, equation (22), for $F_n$, we could use matrix methods to evaluate $Q^n$ and derive the formula. However, knowing Binet’s
formula it is easy to evaluate \( Q^n \) which will lead us to some useful identities. We start by noting that

\[ Q^2 = Q + I, \tag{32} \]

where \( I \) is the identity matrix (This, in fact, is the Cayley-Hamilton theorem for \( Q \)). Since this equation is the same as the equation satisfied by the golden mean, we may immediately conclude from equation (20) that

\[ Q^n = F_nQ + F_{n-1}I = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, \quad n = 1, 2, 3 \ldots \tag{33} \]

Taking the determinant of both sides of this equation yields the identity

\[ (-1)^n = F_{n+1}F_{n-1} - F_n^2, \tag{34} \]

which we will use in discussing a geometric puzzle in the next section.

We obtain another useful identity by using the addition law for matrix powers:

\[ Q^{n+m} = Q^n Q^m, \text{ where } n, m \text{ are nonnegative integers.} \tag{35} \]

If we equate the upper right hand elements of both sides of this equation we find the addition formula for Fibonacci numbers:

\[ F_{n+m} = F_{n-1}F_m + F_nF_{m+1}, \tag{36} \]

an identity we which will be useful in discussing the divisibility properties of Fibonacci numbers. Putting \( n = m \) in the above we get

\[ F_{2n} = F_{n-1}F_n + F_nF_{n+1} = F_n(F_{n+1} + F_{n-1}) \]

\[ F_{2n} = (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1}) \]

\[ F_{2n} = F_{n+1}^2 - F_{n-1}^2. \tag{37} \]

Thus the difference of squares of alternate Fibonacci numbers is again a Fibonacci number.

### 11 A Geometric Puzzle

There is an old geometric ‘proof’ that \( 64 = 65 \). Start with a square of side 8 and subdivide it into four parts as shown in Figure 26. Then put the four parts together to form a rectangle of sides 5 and 13 and area 65 as shown in Figure 27.
Since the area of the square is 64, we have that $64 = 65$. The resolution of the puzzle is that the four pieces do not quite fit together in the rectangle as shown.

If instead of a square of side 8 we take a square of side $F_n$, we may divide it into 4 pieces as show in Figure 28 and reassemble them inside a $F_{n-1}$ by $F_{n+1}$ rectangle as shown in Figure 29. There is a gap (or overlap) in the form of a thin parallelogram. The difference in the area of the square and rectangle is

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1},$$

where we have used the identity given in Equation (34). If $n$ is odd there is parallelogram overlap, while if $n$ is even, there is a parallelogram gap. In any case the difference in areas is always 1, which is small compared to the areas if $n$ is large.
Now let us take a square of side $x + y$, where $x$ and $y$ are any positive real numbers and subdivide it into the four parts shown in Figure 30. We then put the four parts together as shown in the rectangle of sides $x$ and $2x + y$ as shown in Figure 31 and insist that the parts fit together exactly without any gaps or overlaps. This means that

$$(s + y)^2 = x(2x + y).$$

This leads the equation

$$\left(\frac{x}{y}\right)^2 = \left(\frac{x}{y}\right) + 1,$$

so that $x/y = \phi$, the golden mean. Looking again the situation with gaps shown in Figures 28 and 29, if we take $x = F_{n-1}$ and $y = F_{n-2}$, we note that $x/y$ is the ratio of successive Fibonacci numbers, namely $F_{n-1}/F_{n-2}$ which is close to the golden mean if $n$ is large.

12 Divisibility Properties of Fibonacci Numbers

The Fibonacci numbers have some nice divisibility properties. We have listed the first twenty-two Fibonacci numbers together with factored forms for the Fibonacci
Looking at this table we note that if \( n \) is divisible by 3, then \( F_n \) is divisible by \( F_3 \); if \( n \) is divisible by four then \( F_n \) is divisible by \( F_4 \); if \( n \) is divisible by 5, then \( F_n \) is divisible by \( F_5 \). In general we have

**Theorem 1.** If \( n \) is divisible by \( m \) then \( F_n \) is divisible by \( F_m \).

**Proof.** Suppose \( n \) is divisible by \( m \), then \( n = mk \) for some integer \( k \). The proof is by induction on \( k \). Clearly the theorem is trivially true for \( k = 1 \). Assume the theorem is true for an arbitrary \( k \), then we must show it remains true for \( k + 1 \). By the addition formula (36), we have

\[
F_{m(k+1)} = F_{mk-1}F_m + F_{mk}F_{m+1}
\]

By assumption \( F_{mk} \) is divisible by \( F_m \), therefore the right hand side of the above equation is divisible by \( F_m \) and therefore \( F_{m(k+1)} \) is also divisible by \( F_m \). □

At first glance it appears that if \( n > 2 \) is prime, then \( F_n \) is also prime. This holds for \( n = 3, 5, 7, 11, 13, 17 \) but fails for \( n = 19 \) since

\[
F_{19} = 4181 = 37 \cdot 113.
\]

If the conjecture were true it would mean there would be an infinite number of prime Fibonacci numbers. Indeed, it is not presently known whether or not the number of prime Fibonacci numbers is finite or infinite.

However it is true that if \( n \neq 4 \) then if \( F_n \) is a prime then \( n \) is a prime. Equivalently we have:
Theorem 2. If \( n \) is a composite number other than 4, then \( F_n \) is also composite.

Proof. If \( n \geq 6 \) is composite, it must be divisible by some \( m < n \), therefore by Theorem 1, \( F_n \) is divisible by \( F_m \).

Another property that one could guess at from looking at the table is

Theorem 3. Consecutive Fibonacci numbers are relatively prime.

Proof. From equation (34), we have

\[
(-1)^n = F_{n-1}F_{n+1} - F_n^2,
\]

therefore, if \( F_{n+1} \) and \( F_{n+2} \) had a common factor other than 1, it would also have to be a factor of \((-1)^n\), which is impossible.

If \( a, b \) are positive integers we denote the greatest common divisor of \( a \) and \( b \) by \( \gcd(a, b) \) (see the note at the end of this section for the Euclidean algorithm to find \( \gcd(a, b) \)).

Theorem 4. If \( n \) and \( m \) are positive integers then \( \gcd(F_m, F_n) = F_{\gcd(m, n)} \).

Example Take \( m = 12 \) and \( n = 18 \), then \( \gcd(F_{12}, F_{18}) = \gcd(144, 2584) = 8 \) and \( F_{\gcd(12, 18)} = F_6 = 8 \).

Proof. Let \( d = \gcd(m, n) \), then \( d \) divides both \( m \) and \( n \). By Theorem 1, \( F_d \) divides both \( F_m \) and \( F_n \). We must show that \( F_d \) is the greatest common divisor of \( F_m \) and \( F_n \). Since \( d = \gcd(m, n) \), we may use Bezout’s identity to find integers \( r \) and \( s \) so that \( d = mr + ns \) (see the end of this section for a brief explanation of Bezout’s identity). By the addition formula for Fibonacci numbers we have

\[
F_d = F_{mr+ns} = F_{mr-1}F_{ns} + F_{mr}F_{ns+1}.
\]

This shows that any divisor of \( F_m \) and \( F_n \) must also divide \( F_d \); thus \( F_d \) is the greatest common divisor.

Using this theorem we may now prove the converse of Theorem 1.

Theorem 5. If \( F_n \) is divisible by \( F_m \) then \( n \) is divisible by \( m \).

Proof. If \( F_n \) is divisible by \( F_m \) then \( \gcd(F_n, F_m) = F_m \), but according to Theorem 4, we have \( \gcd(F_n, F_m) = F_{\gcd(n, m)} \). Thus \( F_{\gcd(n, m)} = F_m \) or \( m = \gcd(n, m) \), which means \( n \) is divisible by \( m \).
Combining Theorem 1 and Theorem 5 we have

**Theorem 6.** \( F_n \) is divisible by \( F_m \) if and only if \( n \) is divisible by \( m \).

As immediate consequences of this theorem, we have, taking \( m = 2, 3, 5, 7, 11, 13 \):

- Since \( F_3 = 2 \), \( F_n \) is divisible by 2 if and only if \( n \) is divisible by 3
- Since \( F_4 = 3 \), \( F_n \) is divisible by 3 if and only if \( n \) is divisible by 4
- Since \( F_5 = 5 \), \( F_n \) is divisible by 5 if and only if \( n \) is divisible by 5
- Since \( F_7 = 13 \), \( F_n \) is divisible by 13 if and only if \( n \) is divisible by 7
- Since \( F_{11} = 89 \), \( F_n \) is divisible by 89 if and only if \( n \) is divisible by 11
- Since \( F_{13} = 233 \), \( F_n \) is divisible by 233 if and only if \( n \) is divisible by 13

Note that the values of \( m \) we have chosen yield prime values of \( F_m \). If we take \( m = 6 \), and \( F_6 = 8 \) it is still true that \( F_n \) is divisible by 8 if and only if \( n \) is divisible by 6, however we can also show that:

- \( F_n \) is divisible by 4 if and only if \( n \) is divisible by 6.

**Proof**

Replacing \( m + n \) by \( k \) in the addition formular (36), \( F_{n+m} = F_{n-1}F_m + F_nF_{m+1} \), we obtain the useful identity

\[
F_n = F_kF_{n-k+1} + F_{k-1}F_{n-k} \quad \text{(38)}
\]

Putting \( k = 6 \) we find

\[
F_n = 8F_{n-5} + 5F_{n-6},
\]

thus \( F_n \) is divisible by 4 if and only if \( F_{n-6} \) is divisible by 4. Since \( F_0 = 0 \), we see that \( F_{6j} \) is divisible by 4, for \( j = 1, 2, \ldots \). Since \( F_1 = 1 \) is not divisible by 4, neither is \( F_{6j+1} \), for \( j = 1, 2, \ldots \). In a similar manner since \( F_2 = 1, F_3 = 2, F_4 = 3, F_5 \) are not divisible by 4, it follows that \( F_{6j+2}, F_{6j+3}, F_{6j+4}, F_{6j+5} \) are not divisible by 4.  

Using this same approach one can prove:

- \( F_n \) is divisible by 7 if and only if \( n \) is divisible by 8.
- \( F_n \) is divisible by 11 if and only if \( n \) is divisible by 10.
- \( F_n \) is divisible by 6 if and only if \( n \) is divisible by 12.
- \( F_n \) is divisible by 13 if and only if \( n \) is divisible by 14.
- \( F_n \) is divisible by 10 if and only if \( n \) is divisible by 15.
Theorem 6 may be used to give an interesting proof of the well known

**Theorem 7.** There are an infinite number of prime numbers.

**Proof.** This proof is due to M. Wunderlich ([11]). Suppose there are only a finite number of primes. Let \( p_1, p_2, \ldots, p_n \) be a listing of all the primes. Consider

\[
F_{p_1}, F_{p_2}, \ldots, F_{p_n}.
\]

By Theorem 6, these numbers are pairwise relatively prime. Since there are only \( n \) primes, none of these numbers can have more than one prime factor. But this contradicts the fact that \( F_{19} = 3181 = 113 \cdot 37 \).

\[ \square \]

**A note on the Euclidean algorithm and Bezout’s identity.**

The Euclidean algorithm is a method of finding the greatest common divisor of a pair of positive integers \( (a, b) \). Assume that \( b < a \), then we may divide \( a \) by \( b \) to get a quotient \( r_1 \) with \( 0 \leq r_1 < b \), or \( a = bq_1 + r_1 \). It is clear that every divisor of \( a \) and \( b \) is a divisor of \( b \) and \( r_1 \) and vice-versa so that so that \( \gcd(a, b) = \gcd(b, r_1) \). We may repeat this procedure using the pair \( (b, r_1) \) producing a pair \( (r_1, r_2) \) with \( r_2 < r_1 \) and keep this up until we arrive at \( (r_n, 0) \), then \( r_n \) is the \( \gcd(a, b) \).

**Example** Find \( \gcd(42, 30) \). We find

\[
42 = 30 \cdot 1 + 12, \quad \gcd(42, 30) = \gcd(30, 12),
\]
\[
30 = 12 \cdot 2 + 6, \quad \gcd(30, 12) = \gcd(12, 6),
\]
\[
12 = 6 \cdot 2, \quad \gcd(12, 6) = \gcd(6, 0) = 6,
\]

so that \( \gcd(42, 30) = 6 \). (39)

In general we have

\[
a = bq_1 + r_1, \\
b = r_1q_2 + r_2, \\
r_1 = r_1q_3 + r_3, \\
\vdots \\
r_{n-2} = r_{n-1}q_n + r_n, \\
r_{n-1} = r_nq_{n+1}
\]

Bezout’s theorem states that if \( a \) and \( b \) are positive integers then there exists integers \( r \) and \( s \) so that \( \gcd(a, b) = ar + bs \). This follows from the Euclidean
algorithm. Start with the equation for \( r_n = r_{n-2} - r_{n-1}q_n \) substitute for \( r_{n-1} \) from the previous equation, and work backwards through the equations to find integers \( r \) and \( s \) so that \( \gcd(a, b) = ar + bs \).

Example Looking at the previous example we have

\[
\gcd(42, 30) = 6 = 30 - 12 \cdot 2 = 30 - (42 - 30)2 = 42(-2) + 30 \cdot 3.
\]

13 Fibonacci Numbers and the Pascal Triangle

The Fibonacci numbers are closely related to the binomial coefficients, i.e, the numbers

\[
\binom{n}{j} = \frac{n!}{(n-j)! \cdot j!}, \quad 0 \leq j \leq n,
\]

the coefficient of \( x^j \) in the expansion of \((1+x)^n\). These coefficients are often arranged in a Pascal triangle with the \( n \)-th row of the triangle containing the coefficients \( \binom{n}{j} \) for \( 0 \leq j \leq n \). The non-zero elements in the table below represent the first 8 rows of the Pascal triangle, starting with the 0-th row. It is convenient to define

\[
\binom{n}{j} = 0, \quad j > n. \tag{40}
\]

This fills out the Pascal array with zeros as shown below.

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<th>( n )</th>
<th>( \binom{n}{0} )</th>
<th>( \binom{n}{1} )</th>
<th>( \binom{n}{2} )</th>
<th>( \binom{n}{3} )</th>
<th>( \binom{n}{4} )</th>
<th>( \binom{n}{5} )</th>
<th>( \binom{n}{6} )</th>
<th>( \binom{n}{7} )</th>
</tr>
</thead>
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<td>35</td>
<td>35</td>
<td>21</td>
<td>7</td>
<td>1</td>
</tr>
</tbody>
</table>

The Pascal Array
The elements in the $n$-th row of the Pascal array may be obtained from those in the $(n - 1)$-st row using the Pascal recurrence formula
\[
\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}, \quad n, j \geq 1.
\] (41)

Note that the sums of the numbers on the northeast diagonals (rising at $45^\circ$) are the Fibonacci numbers. Here are the first 8 diagonals:

- $1 = \binom{0}{0} = F_1$, $1 = \binom{1}{0} = F_2$
- $1 + 1 = 2 = \binom{2}{0} + \binom{1}{1} = F_3$
- $1 + 2 = 3 = \binom{3}{0} + \binom{2}{1} = F_4$
- $1 + 3 + 1 = 5 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = F_5$
- $1 + 4 + 6 = 3 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2} = F_6$
- $1 + 5 + 6 + 1 = 13 = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = F_7$
- $1 + 6 + 10 + 4 = 21 = \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3} = F_8$

If this pattern continues we would have
\[
F_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots.
\] (42)

Recalling that $\binom{n}{j} = 0$, $j > n$, we may write equation (42) as
\[
F_n = \sum_{j=0}^{n} \binom{n-j-1}{j}, \quad j = 0, 1, 2, \ldots, n-1.
\] (43)

To prove Equation (43) we use induction. Recall that $F_0 = 1$ and $F_1=1$, so equation (43) is true for $n = 0$ and $n = 1$. Assume it is true for $n - 1$ and $n - 2$ that is
\[
F_{n-1} = \sum_{j=0}^{n-1} \binom{n-1-j}{j}, \quad \text{and} \quad F_{n-2} = \sum_{j=0}^{n-2} \binom{n-2-j}{j}.
\]
We must show that Equation (43) holds for $n+1$. We sum both sides of the Pascal recursion formula (41)
\[
\binom{n-j}{j} = \binom{n-j-1}{j-1} + \binom{n-j-1}{j},
\]
to find
\[
\sum_{0}^{n} \binom{n-j}{j} = \sum_{0}^{n} \binom{n-j-1}{j-1} + \sum_{0}^{n} \binom{n-j-1}{j}
\]
\[
= \sum_{0}^{n-2} \binom{n-j-2}{j} + \sum_{0}^{n-1} \binom{n-j-1}{j}
\]
\[
= F_{n-1} + F_{n} = F_{n+1}.
\]

The individual terms in the identity 43 have an interesting interpretation in terms of the original Rabbit problem of section 7. Let

\[ S(n, j) = \text{number of pairs of } k^{th} \text{ generation rabbits at the end of } n \text{ months.} \quad (44) \]

Here the initial pair of rabbits is called the zero-eth generation, the immediate offspring of the initial pair are called first generation rabbits, the immediate offspring of the first generation rabbits are called second generation rabbits, and so on. Our theorem is:

**Theorem 8.**

\[ S(n, j) = \binom{n-j-1}{j}, \quad k = 0, 1, \ldots, n - 1. \quad (45) \]

**Proof.** The proof is based on ([10]). We have the simple accounting equation

\[ S(n, j) = S(n - 1, j) + S(n - 2, j - 1). \quad (46) \]

This merely states that the number of $j^{th}$ generation pairs at the end of the $n^{th}$ month is equal to the number of such pairs at the end of the $(n - 1)^{st}$ month plus the births of $j^{th}$ generation rabbits during the $n^{th}$ months However, the births of $j^{th}$ generation rabbits during the nth month must come from $(j - 1)^{th}$ generation rabbits who are at least two months old; there are precisely $S(n - 2, j - 1)$ such pairs. Since there is only one zeroeth generation pair, we must have

\[ S(n, 0) = 1. \quad (47) \]
To complete the proof, it is necessary only to verify that equation (45) satisfies (46) and (47). Putting $k = 0$ in (45) we find $S(n,0) = 1$. Substituting (45) into (46) we obtain

$$\binom{n-j-1}{j} = \binom{n-j-2}{j} + \binom{n-j-2}{j-1},$$

which is simply the Pascal recursion, equation (41).

Thus, among the 144 pairs of rabbits at the end of 12 months, there are, in addition to the initial pair, 10 first generation, 36 second generation, 56 third generation, 35 fourth generation, and 6 fifth generation pairs.

Another interesting identity involving the Fibonacci numbers and the binomial coefficients is

$$\sum \binom{n}{j} F_j = F_{2n}. \tag{48}$$

To prove this we recall that

$$\sum_0^n \binom{n}{i} x^i = (1+x)^n.$$

using Binet’s formula we obtain

$$\begin{align*}
\sum \binom{n}{j} F_j &= \sum_0^n \binom{n}{j} \left( \frac{\phi^j - \psi^j}{\sqrt{5}} \right) \\
&= \frac{1}{\sqrt{5}} \left( (1+\phi)^n - (1+\psi)^n \right) \\
&= \frac{1}{\sqrt{5}} \left( \phi^{2n} - \psi^{2n} \right) \\
&= F_{2n}.
\end{align*} \tag{49}$$

14 A Fibonacci Search

An important problem in numerical analysis is to locate the point at which a function achieves its minimum or maximum value to within some desired degree of accuracy. Derivatives are sometimes inconvenient or impossible to use and some sort of search procedure must be adopted. Since evaluation of functions is expensive it is desirable to attain the desired degree of accuracy using the smallest
number of function evaluations. For some strange reason Fibonacci numbers pop up here.

We restrict our discussion to a ‘bowl-like’ function, that is, a function defined on an interval \([a, b]\) which is strictly decreasing on \([a, m]\) and strictly increasing on \([m, b]\), for some number \(m\) in the interval. We shall prove:

**Theorem 9.** If \(g(x)\) is bowl-like on \([a, b]\), the point \(m\) where \(g(x)\) achieves its minimum can be located within an interval of length \((b - q)/F_{n+1}\) using at most \(n\) function evaluations and comparisons.

**Proof** Without loss of generality we may assume that \(g(x)\) is defined on the interval \([0, F_{n+1}]\), and show that we may locate \(m\) within a unit interval. We shall show that by using intervals whose lengths are successively smaller Fibonacci numbers. At each stage, we have an interval of length \(F_k\), and the function evaluations occur (except for the last one) at interior points \(p\) and \(q\) whose distances from the ends of the interval are \(F_k - 1\), as shown in the diagram (32). Since \(F_k = 2F_{k-2} + F_{k-3}\), the middle interval has length \(F_{k-1}\).

![Figure 32: A Typical Stage in the Search](image)

We start by evaluating \(g(x)\) at the two interior points \(x_1 = F_{n-1}\) and \(x_2 = F_n\). Three cases may arise: (a) if \(g(x_1) > g(x_2)\), then \(m\) must lie in the interval \(F_{n-1}, F_{n+1}\); (b) if \(g(x_1) < g(x_2)\), then \(m\) must lie in the interval \(0, F_n\); (c) if \(g(x_1) = g(x_2)\), pick either of the two intervals, even though we know \(m\) must lie in \([x_1, x_2]\). In all cases we are left with an interval of length \(F_n\) and the value of \(g(x)\)

![Figure 33: Case (a) and Case (b)](image)
on the left in Figure (34). By evaluating $g(x)$ at $x_3$, we may locate $m$ to within an interval of length $F_{n-1}$. We keep this up until we have reached an interval of length $F_3 = 2$. At this point we have the situation shown on the right in Figure (34) with $g(x)$ known at the midpoint. The last evaluation is at a point $x_n$ close to the midpoint. This allows us to determine which of the two unit intervals contains the minimum $m$.

This procedure is optimal in the sense that one cannot guarantee greater accuracy with no more than $n$ function evaluations; this is demonstrated in Applied Dynamic Programming by Bellman and Dreyfus (Princeton University Press, 1962).

*Example* We take a function $g(x)$, shown in Figure (35), defined on an interval of length $13 = F_7$, and we show we can determine the minimum to within one unit with 6 function evaluations. At the first step we evaluate $g(5)$ and $g(8)$. Since $g(5) > g(8)$, we know that $m$ lies in $[5, 13]$. Referring to Figure (36) we see in the second step, (a), that $g(10) > g(8)$ so that $m$ lies in $[5, 10]$. In the third step, (b) we have $g(7) > g(8)$ so that $m$ must lie in $[7, 10]$. In the fourth step (d) we
have \( g(9) > g(8) \) so that \( m \) must lie in \([7,9]\) in the last step (d), we calculate that \( g(8.01) < g(8) \) so that \( m \) must lie in the interval 8,9. We notice that there are exactly 6 function evaluations, at 5, 8, 10, 7, 9, and 8.01.

![Figure 36: (a) Second Step. (b) Third Step (c) Fourth Step, (d) Final Step](image)

15 History and Lore of the Golden Mean and Fibonacci Numbers

The ancient Greeks, in the fourth and fifth century BC, were aware of the division of a line segment into 'extreme and mean ratio' and it's connections with geometric constructions. The Greeks, however, did not use any special name for this ratio; the names golden mean, golden section and divine proportion were to come later. It is often claimed that the overall dimensions of the Parthenon, are approximately in the golden mean. In about 1909 the symbol \( \phi \)\(^3 \) golden mean was suggested by the American mathematician Mark Barr (see [1], p.420) to honor the Greek sculptor Phidias who, it is claimed, used the golden section in proportioning his sculptures. Going back farther, it is often claimed that the golden mean was used in the construction of the pyramids. The excellent articles by Markowsky ([7]) and Falbo ([5]) debunk these and many other claims.

The Pythagoreans, in the 6-th century BC, were aware of the various golden mean proportion in the pentagon and pentagraph. In fact they selected the five-pointed star as a symbol of their brotherhood.

In 1509, Fra Luca Pacioli elevated the 'extreme and mean ratio' to the 'law of divine proportion' in his book “Divina Proportione” which was illustrated by

\(^3\)This is now common in most works on the golden mean for general audiences. However mathematicians do not always use it. For instance Coxeter ([2], pp. 160-172) uses \( \tau \) for the golden mean
Leonardo da Vinci (see Figure 37). The titles of some of the chapters indicate the reverence Pacioli had for the golden mean: The First Considerable Effect, The Second Essential Effect, The Third Singular Effect, and so on. "The Seventh Inestimable effect" is that the circumradius and the side of a regular pentagon are in the golden ratio. "The Ninth Most Excellent Effect" is that the crossing diagonals on a regular pentagon divide each other in the golden ratio. "The Twelfth Almost Incomprehensible Effect" is that the vertices of three mutually perpendicular golden rectangles are the vertices of a regular icosahedron. "The Thirteenth Most Distinguished Effect" concerns the regular dodecahedron. Pacioli ended his ‘Effects’ at thirteen ‘for the sake of our salvation. Da Vinci’s involvement in Pacioli’s work has led some to claim that Da Vinci used the golden mean in some of his paintings.

Later in the 16-th century P. Rams associated the three parts of the golden section with the trinity and Clavius referred to its ‘god-like proportions.” Kepler, a confirmed mystic, held the ‘divine proportion’ in high esteem. In his own words “Geometry has two great treasures, on the Theorem of Pythagoras, and the other is the division of a line into extreme and mean ratio; the first we may compare to a measure of gold, the second we may name a precious jewel.”

The connection between the golden mean and Fibonacci numbers was made by Kepler in 1611 when he observed that the ratio of successive terms of the Fibonacci sequence approximated the golden mean; this was over 400 years after Fibonacci introduced the numbers. Over a century later, in 1753, R. Simpson that the $F_{n+1}/F_n$ was the $n-th$ convergent in the continued fraction of the golden
mean. Simpson also showed that \( F_{n-1} F_{n+1} - F_n^2 = (-1)^n \).

The general formula for the \( n \)-th Fibonacci number was found in 1843 by J. Binet. A year later Lame’ made use of the Fibonacci numbers in determining the upper limit on the number of operations needed to find the greatest common divisor of two integers. E. Lucas, during the period 1872-1891, made many important contributions to recursive series. He derived the formula connecting Fibonacci numbers to the binomial coefficients and made use of Fibonacci numbers in number theory. Lucas was the first to attach the name Fibonacci to the sequence \( 1, 1, 2, 3, 5, 8, 13, \ldots \). The geometric puzzle of section (11, sometimes called the checkerboard paradox, was discovered by the Sam Lloyd, a famous puzzle author, in 1868.

In the nineteenth century, in numerous writing A. Zeising claimed ‘Der Golden Schnitt’ to be the universal key to beauty in nature and art. Zeising’s theories attracted the psychologist G. Fechner who spent a lot of effort in the attempt to empirically that there is a general intuitive preference for objects which embody a principle of proportion based on the golden mean. This is evidently the source of the often quoted claim that a golden rectangle is the most pleasing to the eye of any rectangle. Falbo ([5]) has given a convincing debunking of these ideas. Modern artists have from time to time been attracted the golden mean as a principle of composition. About 1912, a splinter group of the Cubist movement was formed with the name “La Section d’Or”. The French architect Le Corbusier’s efforts at more harmonious design was based on the golden mean, which he believed to be exemplified in the ideal proportions of the human body as shown in Figure (38). An American, Jay Hambridge, developed a set of principles of design which he called “Dynamic Symmetry”, where proportions, including the golden mean, played an important role. Nowadays, none of the efforts to reduce beauty to a simple mathematical formula is taken seriously.

Defining “beauty” or “ideal proportions” is a subject that probably should be left to philosophers, artists, architects or psychologists. Ralph Waldo Emerson ([4], p. 432) made a pretty good attempt at a definition of beauty:

\[
\text{We ascribe beauty to that which is simple; which has no superfluous parts; which exactly meets its end; which stands related to all things; which is the mean of many extremes.}
\]

Staying strictly within mathematics, the golden mean seems to come close to satisfying this definition. It certainly is simple, is related to many (certainly not all) mathematical things and is the extreme and mean ratio. So, in this sense, we con-
Figure 38: “Ideal” Proportions of the Human Body: \( \frac{AT}{TB} = \frac{A'T'}{T'B'} = \frac{A''T''}{T''B''} = \phi \).

sider the golden mean to have mathematical beauty. But we resist applying this idea to physical, artistic, or architectural beauty.

In 1839 Braun discovered that Fibonacci numbers occurred in connection with the growth of plants. The arrangement of leaves around the twig of a tree is called phyllotaxis. In the elm tree, the leaves appear alternately on both sides of the twig; this is called 1/2 phyllotaxis. In the beach tree, the passage from one leaf to the next involves 1/3 or a turn or 1/3 phyllotaxis. The oak tree exhibits 1/5 phyllotaxis and the pear tree exhibits 3/8 phyllotaxis. Notice all of these are ratios of alternate Fibonacci numbers.

Fibonacci numbers also occur in the arrangement of the scales on a pine cone and the florets in a sunflower. These scales lie in spiral or helical sworls and, in many cases, the rationof right-hand whorls to left-handed whorls are successive Fibonacci numbers.

There have been attempts to come up with some kind of physical laws to explain Fibonacci phyllotaxis. However, there are many exceptions to the Fibonacci phyllotaxis, even within the same plant, so that these attempts seem doomed to failure. For more information on these and other related matters, the reader is

For a wealth of information dealing with Fibonacci numbers, the golden mean and their applications and generalizations, the reader is referred to the issues of the journal *The Fibonacci Quarterly*, established in 1963. It should come as no surprise that the logo of this journal is the pentagram. The Fibonacci Quarterly contains serious mathematics and none of the metaphysical hype. Of the literally hundreds of books about the golden mean and Fibonacci Numbers we mention the ones by Livio ([6]) and Posamentier and Lehmann ([9]).

**References**


[4] Ralph Waldo Emerson. *The Complete Works of Ralph Waldo Emerson: Comprising His Essays Comprising His Essays, Lectures, Poems, and Ora-


