Euler’s Method

- We are interested in knowing not just the qualitative behavior of the solution, but also solving the differential equation quantitatively. Numerical methods are the necessary tools.

- The simplest numerical method,
  - If \( y(x) \) is a solution, then
    \[
y(x_n + h) = y(x_n) + hy'(x_n) + O(h^2)
    \]
    is the Taylor series expansion of \( y(x_n + h) \) near \( x_n \).
  - Suppose the accepted solution at \( x_n \) is given \( y(x_n) \approx y_n \), then the truncated Taylor series suggests that
    \[
y(x_n + h) \approx y_{n+1} = y_n + hf(x_n, y_n)
    \]
    should be a reasonable approximation.

- Questions always asked in numerical ODE:
  - What is the magnitude of the \textit{global error}
    \[
e_n := y_n - y(x_n)
    \]
    at the \( n \)-th step? How does the error propagate?
  - How does the step size \( h \) affect the accuracy?
  - What kinds of errors are involved in the calculation? How do they affect the overall accuracy? How to control the error to get the best possible accuracy?
The local truncation error is defined to be the difference between the exact solution $y(x_{n+1})$ and the approximate solution $y_{n+1}$, provided no previous errors have been introduced into the numerical scheme.

- For Euler's method, the LTE is
  \[ T_{n+1} := y(x_{n+1}) - y(x_n) - hf(x_n, y(x_n)). \] (3)

- Suppose $y''(x)$ is continuous and is bounded by $C$ Then
  \[ T_{n+1} = \frac{h^2}{2} |y''(\xi)| \leq \frac{C}{2} h^2. \]

The (global) errors produced at the previous step will be passed on to the next step. To see how the errors are propagated, subtract (??) from (??). The global error at the next step is given by

\[ e_{n+1} = e_n + h[f(x_n, y_n) - f(x_n, y(x_n))] - T_{n+1} \] (4)

where we have assumed that $f$ satisfies the Lipschitz condition in $y$ with constant $L$.

- It follows that
  \[ |e_{n+1}| \leq (1 + hL)|e_n| + T \] (5)
  where $T = \max |T_n| = O(h^2)$.

- Note that the growth factor $1 + hL$ determines how $e_n$ gets propagated.

- The formula can be applied repeatedly to give
  \[ |e_n| \leq T \left( \frac{(1 + hL)^n - 1}{hL} \right) + (1 + hL)^n|e_0| \] (6)

Look at the example $y' = \lambda y$.

\[ e_{n+1} = (1 + \lambda h)e_n + [-y(x_{n+1}) + (1 + \lambda h)y(x_n)] \]

= Propagated Error + Local Truncation Error.