Deriving the Gaussian Quadrature

- We are interested in deriving quadratures with higher degrees of precision. Toward this end, observe the following three facts:

  ◦ *If a quadrature formula using n distinct abscissas degree of precision ≥ n − 1, then it must result from the integration of an interpolation polynomial.*
  
  ◦ Suppose the quadrature rule is $Q_n(f) = \sum_{i=1}^{n} \alpha_i f(x_i)$.
  
  ◦ Then $\sum_{i=1}^{n} \alpha_i x_i^k = \frac{b^{k+1} - a^{k+1}}{k+1}$ for $k = 0, \ldots, n - 1$.
  
  ◦ We may rewrite this system of equations as
    
    $$
    \begin{bmatrix}
    1 & \ldots & 1 \\
    x_1 & \ldots & x_n \\
    \vdots \\
    x_1^{n-1} & \ldots & x_n^{n-1}
    \end{bmatrix}
    \begin{bmatrix}
    \alpha_1 \\
    \alpha_2 \\
    \vdots \\
    \alpha_n
    \end{bmatrix}
    =
    \begin{bmatrix}
    b - a \\
    \frac{b^2 - a^2}{2} \\
    \vdots \\
    \frac{b^n - a^n}{n}
    \end{bmatrix}.
    $$
    \hspace{1cm} (1)

  The coefficient matrix is the Vandermonde matrix. Thus THE system (1) has a unique solution for $\alpha_i, i = 1, \ldots, n$.
  
  ◦ On the other hand, we can construct an interpolating polynomial of degree $n - 1$ using the same nodes $\{x_i\}_{i=1}^{n}$.
  
  ◦ This polynomial results in a quadrature formula $\sum_{i=1}^{n} \beta_i f(x_i)$ which has degree of precision at least $n - 1$.
  
  ◦ By setting $E_n(x^k) = 0$ for $k = 0, \ldots, n - 1$ in the new quadrature, we end up with the same linear system (1).
  
  ◦ By uniqueness, it must be that $\alpha_i = \beta_i$. 

Since the Gaussian quadrature is expected to have degree of precision higher than $n$, we conclude that

- It may be thought of as the integration of a certain polynomial that interpolates $f(x)$ at a certain set of nodes $x_1, \ldots, x_n$.
- It must remain true that

$$E_n(f) = \int_a^b f[x_1, \ldots, x_n, t] \prod_{i=1}^n (t - x_i) dt. \quad (2)$$

- Not only $E_n(x^k) = 0$ for $k = 0, \ldots, n-1$, but we further would like to require $E_n(x^k) = 0$ for $k = n, \ldots, n+\nu$, and for $\nu$ as large as possible.

If $f(t) = t^{n+\nu}$, $\nu \geq 0$, then the $n$-th divided difference $f[x_1, \ldots, x_n, t]$ is a polynomial of degree at most $\nu$.

- When $n = 1$, we find $f[x_1, t] = \frac{f(x_1) - f(t)}{x_1 - t}$ is obviously a polynomial of degree $\nu$.
- Suppose the assertion is true for $n = k$. Consider $f(t) = t^{k+1+\nu}$. (We are preparing to use the Induction Principle on $n$.)
- Regard $f(t) = t^{k+1+\nu}$. Then, by induction hypothesis, the difference quotient $f[x_2, \ldots, x_{k+1}, t]$ is a polynomial of degree at most $\nu + 1$.
- Observe that

$$f[x_1, \ldots, x_{k+1}, t] = \frac{f[x_1, \ldots, x_{k+1}] - f[x_2, \ldots, x_{k+1}, t]}{x_1 - t}.$$  

Note that the numerator has a zero at $t = x_1$. After cancelation, $f[x_1, \ldots, x_{k+1}, t]$ is a polynomial of degree of most $\nu$.
- The assertion now follows from the induction.

- With all said, if we choose the nodes $x_i$ so that $\omega(t) = \prod_{i=1}^n (t - x_i)$ is perpendicular to all lower degree polynomials, then the error $E(x^{n+\nu})$ would be zero for $\nu = 0, 1, \ldots n - 1$. The theory of orthogonal polynomials now kicks in.