Error in Interpolation

- Suppose the polynomial $p(t)$ interpolates a function $f(t)$ at nodes $t = x_0, x_1, \ldots, x_n$, i.e., suppose $p(x_i) = f_i = f(x_i)$ for all $i = 0, 1, \ldots, n$. Define
  $$e(t) = f(t) - p(t)$$
as the error function. What can be said about the behavior of $e(t)$?

  ◦ Choose $x_{-1} \neq x_i$ for any $i = 0, 1, \ldots, n$. Define
    $$F(u) = f(u) - p(u) - (f(x_{-1}) - p(x_{-1}))(n+1)\prod_{i=0}^n\frac{(x - x_i)}{x_{-1} - x_i}.$$   

  ◦ Observe $F(x_i) = 0$ for $i = -1, 0, 1, \ldots, n$, i.e., $F(u)$ has $n + 2$ zeros.
  ◦ By Rolle’s theorem, there exists $\xi$ between $x_{-1}, x_0, \ldots, x_n$ such that $F^{(n+1)}(\xi) = 0$.
  ◦ Note that
    $$f^{(n+1)}(\xi) - (f(x_{-1}) - p(x_{-1}))\frac{(n + 1)!}{\prod_{i=0}^n(x_{-1} - x_i)}.$$   

  ◦ Since $x_{-1}$ is arbitrary, we have established
    $$e(t) = \frac{\prod_{i=0}^n(t - x_i)}{(n + 1)!}f^{(n+1)}(\xi)$$ (1)

for some $\xi$ between $t, x_0, \ldots, x_n$.

   ◦ Note that $\xi = \xi(t)$ varies as $t$ varies.