Polynomial Evaluation

- Given a polynomial in the natural form
  
  \[ p(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0, \]

  the evaluation of \( \hat{p} = p(\hat{t}) \) can be done stably by an algorithm called synthetic division:

  \[
  p = a[n] \\
  \text{for } i = n-1:-1:0 \\
  \quad p = p*t + a_{-}[i] \\
  \text{end} \\
  \]

  - Synthetic division requires only \( n \) additions and \( n \) multiplications.
  - It is quite efficient.
  - Synthetic division is only quite stable in the sense that the computed value of \( p(t) \) is the exact value of a polynomial \( \tilde{p} \) whose coefficients differ from those of \( p \) by relative errors on the order of the rounding unit. (Students! Read this statement one more time. This is normally what is meant by stability shown by backward error analysis.)
  - Note that the Lagrange polynomials are not in the natural form and hence is difficult to evaluate.

- We say a polynomial \( p(t) \) is the Newton form if

  \[ p(t) = c_0 + c_1(t-x_0) + c_2(t-x_0)(t-x_1) + \ldots + c_n(t-x_0)(t-x_1) \ldots (t-x_{n-1}). \]

  - Evaluation of a Newton form is easy:

    \[
    p = c[n] \\
    \text{for } i = n-1:-1:0 \\
    \quad p = p*(t-x[i]) + c[i] \\
    \text{end} \\
    \]

  - It remains to determine the coefficients \( c_0, \ldots, c_n \) so that \( p(t) \) interpolates the data \( \{(x_i, f_i)\} \) for \( i = 0, 1, \ldots, n. \)
Determining the Newton Form

- The coefficients of the Newton form of an interpolant can be determined through the system

\[
\begin{align*}
  f_0 &= c_0 \\
  f_1 &= c_0 + c_1(x_1 - x_0) \\
  &\vdots \\
  f_n &= c_0 + c_1(x_n - x_0) + \ldots + c_n(x_n - x_0) \ldots (x_n - x_{n-1}).
\end{align*}
\]

\[\begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
1(x_1 - x_0) & 0 & \ldots & 0 \\
1(x_2 - x_0)(x_2 - x_0)(x_2 - x_1) & \ldots & 0 \\
& \vdots & \ddots & \vdots \\
1(x_n - x_0)(x_n - x_0)(x_n - x_1) & \ldots & (x_n - x_0) \ldots (x_n - x_{n-1})
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix}
= 
\begin{bmatrix}
f_0 \\
f_1 \\
f_2 \\
\vdots \\
f_n
\end{bmatrix}.\]

- If new points are added to be interpolated, the coefficients already determined will \textit{not} be affected. We just need to add a new row to determine \(c_{n+1}\).

- There is a yet better way, called the Newton divided differences, to determine the coefficients.