On the Differential Equation
\[ \frac{dX}{dt} = [X, k(X)] \]
where \( k \) is a Toeplitz Annihilator

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Inverse Eigenvalue Problem (IEP)

• Given
  – Real symmetric matrices $A_0, A_1, \ldots, A_n \in \mathbb{R}^{n \times n}$;
  – Real numbers $\lambda_1^* \geq \ldots \geq \lambda_n^*$;

• Find
  – Values of $c := (c_1, \ldots, c_n)^T \in \mathbb{R}^n$,

• Such that
  – Eigenvalues of the matrix
    \[ A(c) := A_0 + c_1 A_1 + \ldots + c_n A_n \]
    are precisely $\lambda_1^*, \ldots, \lambda_n^*$.
Existence Question

• Not always will the (IEP) have a solution.

• Inverse Toeplitz Eigenvalue Problem (ITEP):
  
  – A special case of the (IEP) where $A_0 = 0$ and $A_k := (A_{ij}^{(k)})$ with

  $$ A_{ij}^{(k)} := \begin{cases} 
    1, & \text{if } |i - j| = k - 1; \\
    0, & \text{otherwise.}
  \end{cases} $$

  – Existence question for (ITEP) remains open for $n \geq 5$. 
Notation

• $S(n) :=$ The subspace of all symmetric matrices in $\mathbb{R}^{n \times n}$.
• $O(n) :=$ The manifold of all orthogonal matrices in $\mathbb{R}^{n \times n}$.
• $T :=$ The subspace of all Toeplitz matrices in $S(n)$.
• $\Lambda := \text{diag}\{\lambda_1, \ldots, \lambda_n\}$.
• $\mathcal{M}(\Lambda) := \{Q\Lambda Q^T \mid Q \in O(n)\}$
  - Contains all matrices in $S(n)$ whose eigenvalues are precisely $\lambda_1^*, \ldots, \lambda_n^*$.
• $\mathcal{A} := \{A(c) \mid c \in \mathbb{R}^n\}$.
  - Solving the (IEP) is equivalent to finding an intersection of the two sets $\mathcal{M}(\Lambda)$ and $\mathcal{A}$. 
A Descent Method for ITP

• Assume
  – Matrices $A_1, \ldots, A_n$ are mutually orthonormal.
  – Matrix $A_0$ is perpendicular to all $A_k$, for $k = 1, \ldots, n$.

• The distance from $X$ to the affine subspace $\mathcal{A}$ is

$$dist(X, \mathcal{A}) = \|X - (A_0 + P(X))\|$$

where

$$P(X) = \sum_{k=1}^{n} <X, A_k> A_k.$$ 

• Approach the (IEP) by solving the optimization problem:

Minimize $F(Q) := \frac{1}{2}\|Q^T \Lambda Q - A_0 - P(Q^T \Lambda Q)\|^2$

Subject to $Q \in \mathcal{O}(n)$. 
Compute the Projected Gradient

- The gradient $\nabla F$ can be calculated:
  \[
  \nabla F(Q) = 2\Lambda Q\{Q^T\Lambda Q - A_0 - P(Q^T\Lambda Q)\}.
  \]

- Projection is easy because:
  \[
  R^{n \times n} = T_Q\mathcal{O}(n) \oplus T_Q\mathcal{O}(n)^\perp = QS(n)^\perp \oplus QS(n)
  \]

- The vector field
  \[
  \frac{dQ}{dt} = Q[Q^T\Lambda Q, A_0 + P(Q^T\Lambda Q)]
  \]
  where
  \[
  [A, B] := AB - BA
  \]
  defines a steepest descent flow on the manifold $\mathcal{O}(n)$ for the objective function $F(Q)$. 
A Descent Flow for the IEP

• Define

\[ X(t) := Q(t)^T \Lambda Q(t). \]

• \( X(t) \) is governed by:

\[ \frac{dX}{dt} = [X, [X, A_0 + P(X)]]. \]

• Starting with any given \( X(0) \in \mathcal{M}(\Lambda) \), the solution \( X(t) \) of the initial value problem will
  
  – Stay on the surface \( \mathcal{M}(\Lambda) \).
  
  – Move in the steepest descent direction to minimize \( dist(X(t), \mathcal{A}) \).
• Suppose

\[ X(t) \rightarrow \hat{X} \text{ as } t \rightarrow \infty. \]

\[ \hat{X} \text{ is also in } \mathcal{A}. \]

Then \( c_i = < \hat{X}, A_i > \) for \( i = 1, \ldots, n \) is a putative solution to the (IEP).

• UNFORTUNATELY, the flow \( X(t) \) for the (ITEP) sometimes converges to a stable equilibrium point that is not Toeplitz.
A New Approach

• To stay on the surface $\mathcal{M}(\Lambda)$, a differential equation must take the form

$$\frac{dX}{dt} = [X, k(X)]$$

where $k : S(n) \rightarrow S(n)^\perp$.

• Require $k$ to be a linear Toeplitz annihilator:

$-k(X) = 0$ if and only if $X \in \mathcal{T}$.
• What is the idea?
  
  – Suppose all elements in Λ are distinct.
  – \([X, k(X)] = 0\) if and only if \(k(X)\) is a polynomial of \(X\).
  – \(k(X) \in \mathcal{S}(n) \cap \mathcal{S}(n)^\perp = \{0\}\).
  – \(||X(t)|| = ||\Lambda||\) for all \(t \in R\).
  – A bounded flow on a compact set must have a non-empty \(\omega\)-limit set.

• Can such a \(k\) be defined?
  
  – The simplest choice:
    \[
    k_{ij} := \begin{cases} 
      x_{i+1,j} - x_{i,j-1}, & \text{if } 1 \leq i < j \leq n \\
      0, & \text{if } 1 \leq i = j \leq n \\
      x_{i,j-1} - x_{i+1,j}, & \text{if } 1 \leq j < i \leq n 
    \end{cases}
    \]
The Differential Equation

• Only consider
  – The upper triangular part of a matrix.
  – $n = 3$.

• The differential equation is invariant under the translation $X + \sigma I$. Thus
  – Assume
    \[ \lambda_1 + \lambda_2 + \lambda_3 = 0. \]
  – Eliminate one variable
    \[ x_{22} = -x_{11} - x_{33}. \]
• The differential system is equivalent to

\[
\begin{align*}
\frac{dx_{11}}{dt} &= 4x_{12}x_{11} + 2x_{12}x_{33} - 2x_{13}x_{23} + 2x_{13}x_{12}, \\
\frac{dx_{12}}{dt} &= -4x_{11}^2 - 4x_{11}x_{33} - 2x_{13}x_{33} - x_{13}x_{11} \\
&\quad - x_{33}^2 - x_{23}^2 + x_{23}x_{12}, \\
\frac{dx_{13}}{dt} &= 3x_{11}x_{23} + 3x_{12}x_{33}, \\
\frac{dx_{23}}{dt} &= x_{23}x_{12} - x_{12}^2 - 4x_{11}x_{33} - x_{11}^2 - 4x_{33}^2 \\
&\quad - 2x_{13}x_{11} - x_{13}x_{33}, \\
\frac{dx_{33}}{dt} &= 2x_{13}x_{23} - 2x_{13}x_{12} + 4x_{23}x_{33} + 2x_{11}x_{23}.
\end{align*}
\]
Critical Points

- The vector field is a system of homogeneous polynomials of degree 2.
  - No isolated equilibrium for the differential system.

- Only two kinds of real equilibria (by using the theory of Gröbner bases):
  - \((c_1, 0, -3c_1, 0, c_1), c_1 \in \mathbb{R}\)
  - \((0, c_2, c_3, c_2, 0), c_2, c_3 \in \mathbb{R}\)

- Local stability:
  - Eigenvalues at \((c_1, 0, -3c_1, 0, c_1)\) are
    \[0, \pm 3\sqrt{6}c_1, \pm 6\sqrt{2}|c_1|i.\]

  * This kind of equilibrium can never be stable.
  * Near such an equilibrium, there should be periodic solutions.
  * The number of zero eigenvalue indicates the dimension of the manifold of equilibria.
– Eigenvalues at \((0, c_2, c_3, c_2, 0)\) are

\[
0, 0, 6c_2, 2(c_2 + c_3), 2(c_2 - c_3).
\]

* The equilibrium can be stable only if

\[
c_2 < 0 \quad \text{and} \quad |c_2| \geq c_3.
\]

* The equilibrium corresponds to a Toeplitz matrix

\[
\begin{bmatrix}
0 & c_2 & c_3 \\
c_2 & 0 & c_2 \\
c_3 & c_2 & 0
\end{bmatrix}
\]

whose eigenvalues are:

\[
\frac{c_3 - \sqrt{c_3^2 + 8c_2^2}}{2}, -c_3, \frac{c_3 + \sqrt{c_3^2 + 8c_2^2}}{2}
\]

in ascending order.

* The case \(|c_2| = c_3\) corresponds to multiple eigenvalues.
Invariant Sets

- Due to the homogeneity,
  - If $X(t)$ is a solution, so is $Y(t) := \frac{1}{\alpha}X(\frac{t}{\alpha})$ for any real constant $\alpha$.
  - If $\mathcal{I}$ is any set invariant under the differential equation, so is the set $\alpha \mathcal{I}$.

- A matrix is said to be
  - (Skew-) Persymmetric if it is (skew-) symmetric about the NE-SW diagonal.

- The subspace $\mathcal{W}$ of all persymmetric matrices in $\mathcal{S}(n)$ is invariant.
  - $\mathcal{W}$ is a 3-dimensional subspace:
    $$\mathcal{W} = \{(x_{11}, x_{12}, x_{13}, x_{12}, x_{11}) | x_{11}, x_{12}, x_{13} \in \mathbb{R}\}.$$
The intersection of $\mathcal{W}$ and $\mathcal{M}(\Lambda)$ consists of three "ellipses":

\[
(x_{11} - \frac{\lambda_3}{4})^2 + \frac{1}{2}x_{12}^2 = \frac{(2\lambda_1 + \lambda_3)^2}{16},
\]
\[
x_{13} = x_{11} - \lambda_3;
\]

\[
(x_{11} - \frac{\lambda_1}{4})^2 + \frac{1}{2}x_{12}^2 = \frac{(\lambda_1 + 2\lambda_3)^2}{16},
\]
\[
x_{13} = x_{11} - \lambda_1;
\]

\[
(x_{11} + \frac{\lambda_1 + \lambda_3}{4})^2 + \frac{1}{2}x_{12}^2 = \frac{(\lambda_1 - \lambda_3)^2}{16},
\]
\[
x_{13} = x_{11} + \lambda_1 + \lambda_3.
\]

- The projections of these ellipses onto the $(x_{11}, x_{12})$-plane must be such that one circumscribes the other two.

- For $n = 3$ the (ITEP) has exactly
  * Four real solutions if all given eigenvalues are distinct.
  * Two real solutions if one eigenvalue has multiplicity 2.
Flows on $\mathcal{W}$

- The differential equation restricted on $\mathcal{W}$ is given by:
  \[
  \frac{dx_{11}}{dt} = 6x_{11}x_{12},
  \]
  \[
  \frac{dx_{12}}{dt} = -9x_{11}^2 - 3x_{11}x_{13},
  \]
  \[
  \frac{dx_{13}}{dt} = 6x_{11}x_{12}.
  \]

- $\mathcal{W}$ itself consists of layers of 2-dimensional invariant affine subspaces:
  - Each affine subspace is determined by
    \[
    x_{13} = x_{11} + c_4
    \]
    for a certain real constant $c_4$.
  - For any given $c_4$, the integral curves on the invariant affine subspace are determined by
    \[
    \left(x_{11} + \frac{c_4}{4}\right)^2 + \frac{1}{2}x_{12}^2 = c_5^2
    \]
    for real constants $c_5$. 
• These elliptic orbits are concentric with the center \((-\frac{c_4}{4}, 0, \frac{3c_4}{4})\) which is an equilibrium of the first kind.

  – There are periodic solutions near that equilibrium.
  – For large enough \(c_5^2\), a non-periodic solution of will converge as is shown.

    * The limit point corresponds to an equilibrium of the second kind

    \[
    (0, -\sqrt{2c_5^2 - \frac{c_4^2}{8}}, c_4, -\sqrt{2c_5^2 - \frac{c_4^2}{8}}, 0).
    \]

    * The limit point can be stable for the entire system only if

    \[
    c_5^2 \geq \frac{9c_4^2}{16}.
    \]

• For any given \(\{\lambda_1, \lambda_2, \lambda_3\}\), the surface \(\mathcal{M}(\Lambda)\) can have one and only one equilibrium which is stable for the differential system.
Orbital Stability

• The existence of periodic solutions is disappointing.
• A computer plot of $x_{12}(t)$ versus $x_{11}(t)$ for a single trajectory where
  – Initial values $x_{11}(0) = 1.0$, $x_{12}(0) = 1.0$, $x_{13} = -3.0$, $x_{14} = 1.0$, $x_{15} = 1.0$ indicate the true trajectory should be an ellipse.
  – Interval of integration is $0 \leq t \leq 11.05$.
  – A variable-step variable-order numerical method with high accuracy of error control ($\leq 10^{-14}$) fails to stay close to the ellipse.
• Calculate the characteristic exponents of the linearized system:
  – The period is estimated to be $T = 1.04719755120$ (accurate up to the 11-th digit).
  – The fundamental matrix $\Phi(t) = P(t)e^{tR}$ for the linear system is calculated with $\Phi(0) = I$.  

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The corresponding characteristic exponents (eigenvalues of $R$) are estimated to be:

$$\pm 7.8998, \pm 2.0222 \times 10^{-5}, 4.5297 \times 10^{-14}.$$

The first positive characteristic exponent clearly indicates the orbital unstability.

- Although the numerical solution fails to track down the elliptic orbit, it stays close to the surface $\mathcal{M}(\Lambda)$.

- See the plot of the difference of eigenvalues (measured in the 2-norm) between $X(0)$ and $X(t)$. The error is certainly acceptable within machine roundoff.

- Although the numerical solution is meaningless to the original initial value problem, the final false limit point does solve the (ITEP).
More Questions

- Are there any invariant sets other than the ones we have found?

- With $X(0) = \Lambda$, the solution $X(t)$ displays a special feature — diagonals of $X(t)$ alternate symmetry of evenness with oddness. Does this mean anything?

- Starting with $X(0) = \Lambda$, the solution flow has been observed numerically to always converge a symmetric Toeplitz matrix as $t \rightarrow \infty$. How to argue analytically that this is the case?

- How much can the understanding for $n = 3$ be generalized to higher dimensional case?