Minimal Variance Estimator of Reconstructor in Adaptive Optics Systems

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January 9, 1999
Outline

• Imaging through the Atmosphere
• Closed-loop Adaptive Optics Model
• Adaptive Optics Control
  ◊ Estimating the Reconstructor
  ◊ Controlling the Deformable Mirror
• Numerical Challenges
Atmospheric Imaging Computation

- Purpose:
  - To compensate for the degradation of astronomical image quality caused by the effects of atmospheric turbulence.

- Two stages of approach:
  - Partially nullify optical distortions by a deformable mirror (DM) operated from a closed-loop adaptive optics (AO) system.
  - Minimize noise or blur via off-line post-processing deconvolution techniques (not this talk).

- Challenges:
  - Atmospheric turbulence can only be measured adaptively.
  - Need theory to pass atmospheric measurements to knowledge of actuating the DM.
  - Require fast performance of large-scale data processing and computations.
A Simplified AO System
Basic Notation

• Three quantities:
  ◇ $\phi(t) = \text{turbulence-induced phase profile at time } t.$
  ◇ $a(t) = \text{deformable mirror (DM) actuator command at time } t.$
  ◇ $s(t) = \text{wavefront slope sensor (WFS) measurement at time } t \text{ and with no correction.}$

• Two transformations:
  ◇ $H := \text{transformation from actuator commands to resulting phase profile adjustments.}$
  ◇ $G := \text{transformation from actuator commands to slope sensor measurement adjustments.}$
A Close-loop AO Control Model

\[ \Sigma \]

Open-loop Sensor Measurements
\[ s \]

Corrected Sensor Measurement
\[ \hat{s} = G_a \]

Corrected Phase Profile
\[ \hat{\phi} = H_a \]

Open-loop Phase Profile
\[ \phi \]

Closed-loop Sensor Measurements
\[ \Delta s = s - G_a \]

Closed-loop Phase Profile
\[ \Delta \phi = \phi - H_a \]

Actuator Command
\[ a \]

Loop Compensation
Reconstructor
\[ e = M(s - G_a) \]

(Estimated Residual Phase Error)
From Actuator to DM Surface

- $H$ is used to describe the DM surface change due to the application of actuators.
- $r_i(\vec{x}) = \text{influence function on the DM surface at position } \vec{x} \text{ with an unit adjustment to the } i\text{th actuator.}$
- Assuming $m$ actuators and linear response of actuators to the command, model the DM surface by
  
  \[ \hat{\phi}(\vec{x}, t) = \sum_{i=1}^{m} a_i(t) r_i(\vec{x}). \]

  - Sampled at $n$ DM surface positions, can write
    \[ \hat{\phi}(t) = Ha(t) \]
    \[ \Rightarrow H = (r_i(\vec{x}_j)) \in \mathbb{R}^{m \times m}. \]
    \[ \Rightarrow \hat{\phi}(t) = [\hat{\phi}(\vec{x}_1, t), \ldots, \hat{\phi}(\vec{x}_n, t)]^T \in \mathbb{R}^n = \text{discrete corrected phase profile at time } t. \]
From Actuator to WFS Measurement

- $G$ is used to describe the WFS slope measurement associated with the actuator command $a$.

- Consider the H-WFS model where

\[ s_j(t) := -\int d\bar{x}(\nabla W_{s_j}(\bar{x}) \cdot \vec{d}_j)\phi(\bar{x}, t), \quad j = 1, \ldots, \ell. \]

- $W_{s_j}, \vec{d}_j = \text{given specifications of } j \text{th subaperture.}$

- The measurement corresponding to $\hat{\phi}(\bar{x}, t)$ would be

\[ \hat{s}_j(t) = \sum_{i=1}^{m} \left( -\int d\bar{x}(\nabla W_{s_j}(\bar{x}) \cdot \vec{d}_j) r_i(\bar{x}) \right) G_{ji} a_i(t). \]

- Can write

\[ \hat{s}(t) = Ga(t) \]

where $G = [G_{ij}] \in \mathbb{R}^{\ell \times m}$.

- The DM actuators are not capable of producing the exact wavefront phase $\phi(\bar{x}, t)$ due to its finiteness of degrees of freedom. So $\hat{s} = Ga$ is not an exact measurement.
What is Available?

- Two residuals that are available in a *closed-loop* AO system:
  - $\Delta \phi(t) := \phi(t) - Ha(t)$
    - Represents the residual phase error remaining after the AO correction.
    - Also means instantaneous closed-loop wavefront distortion at time $t$.
  - $\Delta s(t) := s(t) - Ga(t)$
    - Represents feedback applied to $s(t)$ by DM actuator adjustment.
    - Also means *observable* wavefront sensor measurement at time $t$.
- In practice, there is a servo lag or delay in time $\Delta t$, i.e., it is likely
  - $\Delta \phi(t) := \phi(t) - Ha(t - \Delta t)$.
  - $\Delta s(t) := s(t) - Ga(t - \Delta t)$.

Thus the data collected are not perfect.
Open-loop Model

- Assume a linear relationship between open-loop WFS measurement $s$ and turbulence-induced phase profile $\phi$:

$$ s = W\phi + \epsilon $$  \hspace{1cm} (1)

- $\epsilon = $ measurement noise with mean zero.
- In the H-WFS model, $W$ represents a quadrature of the integral operator evaluated at designated positions $\tilde{x}_j$, $j = 1, \ldots, n$.

- Want to estimate $\phi$ using $\tilde{\phi}$ from the model

$$ \tilde{\phi} = M_{open}s $$

so that the variance

$$ \mathcal{E}[||\phi - \tilde{\phi}||^2] $$

is minimized.

- The wave front reconstruction matrix $M_{open}$ is given by

$$ M_{open} = \mathcal{E}[\phi s^T](\mathcal{E}[ss^T])^{-1}. $$

- For unbiased estimation, need to enforce the condition that $M_{open}W = I$. 

Closed-loop Model

- For the H-WFS model, it is reasonable to assume the relationship

\[ WH = G. \]  \hspace{1cm} (2)

- Then

\[
\begin{align*}
    s &= W\phi + \epsilon \\
    &= W(Ha + \Delta\phi) + \epsilon \\
    &= WHa + (W\Delta\phi + \epsilon).
\end{align*}
\]

It follows that

\[ \Delta s = W\Delta\phi + \epsilon. \]  \hspace{1cm} (3)

The closed-loop relationship (3) is identical to the open-loop relationship (1).

- Can estimate the residual phase error $\Delta\phi(t)$ using $e(t)$ from the model

\[ e = M_{\text{closed}}\Delta s \]

- $M_{\text{closed}} = \text{wavefront reconstruction matrix.}$
- For unbiased estimation, it requires that $M_{\text{closed}}W = I$. Hence

\[ M_{\text{closed}}G = M_{\text{closed}}(WH) = H. \]
Estimating the Reconstructor and the Actuator Command

A. Compute $M$ based on the control law $a = Ms$ so that

$$E[||\Delta s||^2] = E[||s - Ga||^2]$$

is minimized.

B. Compute $M$ subject to the servo-loop compensator so that

$$E[\langle \Delta \phi, \Delta \phi \rangle]$$

is minimized, and then determine the DM actuator command $A$ from the finite temporal response loop model

$$\frac{da}{dt} = k\Delta s = kM(s - Ga). \quad (4)$$

C. Compute $M$ based on open-loop measurement so that

$$E[||\Delta \phi - M\Delta s||^2]$$

is minimized.
D. Find $a$ such that

$$\mathcal{E}[\|\Delta \phi\|^2] = \mathcal{E}[\|\phi - Ha\|^2]$$

is minimized subject to

$$Ha = M_{\text{open}}s$$

◊ This is equivalent to the idea case when both the minimum variance approximation $\hat{\phi} = M_{\text{open}}s$ and the DM surface $\hat{\phi} = Ha$ is exactly equal to the induced wave front $\phi$. 
Idea A: 

Minimize $\mathcal{E}[\|\Delta s\|^2]$ 

- Consider the model 
  $$s = Ga + \Delta s.$$ 
  
  Want to determine $M$ and the estimated command $\hat{a}$ of the form 
  $$\hat{a} = Ms$$ 
  
  so that 
  $$\mathcal{E}[\|s - G\hat{a}\|^2]$$ 
  is minimized. 

  $\diamond$ The issue is not to minimize $\mathcal{E}[\|Ms - a\|^2]$. 

- The optimal solution is given by 
  $$M = \left(G^T\mathcal{E}[\Delta s(\Delta s)^T]\right)^{-1}G + (\mathcal{E}[aa^T])^{-1}G^T(\mathcal{E}[\Delta s(\Delta s)^T])^{-1}.$$ 

  $\diamond$ If the noise variance matrix $\mathcal{E}[\Delta s(\Delta s)^T] = \sigma^2 I$, then 
  $$M = (G^T G + \sigma^2(\mathcal{E}[aa^T])^{-1})^{-1} G^T$$ 

  which is reduced to the standard least squares solution if noise variance in $\Delta s$ decreases to zero.
Idea B:

Minimize $\mathcal{E}[\langle \Delta \phi, \Delta \phi \rangle]$ with Loop Compensation

- Assume $MG \equiv H$ and $HM = H$. The steady-state solution is given by

$$a(t) = \int_0^\infty e^{-kMG\tau}kMs(t-\tau)\,d\tau$$

$$= M \left( \int_0^\infty e^{-k\tau}kS(t-\tau)\,d\tau \right).$$

$ y(t) $ means temporally filtered version of the instantaneous slope $s(t)$.

- To minimize $\mathcal{E}[\langle \phi, \phi \rangle]$, $M$ must be given by

$$M = H \left( BS^{-1} + (I - BS^{-1}G)(G^T S^{-1}G)^{-1}G^T S^{-1} \right).$$

where

$$B_{ij} := \mathcal{E}[\langle \phi, h_i \rangle y_j]$$

$$S_{ij} := \mathcal{E}[y_i y_j]$$

$$H = [h_1, \ldots, h_n].$$
Idea C:

Minimize $\mathcal{E}[\|\Delta \phi - M \Delta s\|^2]$

- Consider the closed-loop model
  \[
  \Delta s = W \Delta \phi + \epsilon.
  \]
  and the relationship
  \[
  \Delta \phi - M \Delta s = (\phi - Ha) - M(s - Ga)
  = (\phi - Ms) + (MG - H)a.
  \]
- One could minimize the closed-loop system $\mathcal{E}[\|\Delta \phi - M \Delta s\|^2]$ via minimizing the open-loop system
  
  minimize $\mathcal{E}[\|\phi - Ms\|^2]$
  subject to $MG = H$. 
Numerical Challenges

- Need to compute $M$ fast enough.
- Every formulation involves calculating the inverse of some covariance matrices or sum of nested matrices.
  - Noise covariance matrix $(\mathcal{E}[\Delta s(\Delta s)^T])^{-1}$.
  - Control covariance matrix $(\mathcal{E}[aa^T])^{-1}$.
  - Nested matrix $\left(G^T(\mathcal{E}[\Delta s(\Delta s)^T])^{-1}G+(\mathcal{E}[aa^T])^{-1}\right)^{-1}$.
  - Open-loop estimator $M_{open} = \mathcal{E}[\phi s^T](\mathcal{E}[ss^T])^{-1}$.
- Statistical information about $\phi$, $s$, $\Delta s$ and $a$ varies in time and is available only adaptively.
- Could the constructor be estimated adaptively from the optimization problem itself, instead of the closed-form formulation?