Chapter 1

Introduction

- Overview
- A Brief History
- Classification
- A Glimpse of Some Known Results
- Conclusion
Overview

Often a physical process is described by a mathematical model of which parameters represent important physical quantities.

◊ Direct analysis — Analyze or predict the behavior of the underlying physical process from the parameters.

◊ Inverse analysis — Validate, determine, or estimate the parameters adaptively from the behavior of the physical process.
Inverse Eigenvalue Problem (IEP)

• The mathematical model involves matrices whose spectral properties determine the dynamics of the physical system.

• Reconstruct a matrix from prescribed spectral data.
  ◦ Spectral data may involve a mixture information of eigenvalues or eigenvectors.
  ◦ Sometimes complete information is difficult to obtain. Only partial information is available.
  ◦ For feasibility, often necessary to restrict the construction to special classes of matrices.
Fundamental Questions

- Solvability:
  - Determine a necessary or a sufficient condition under which an IEP has a solution.

- Computability:
  - Develop a scheme through which, knowing a priori that the given spectral data are feasible, a matrix can be constructed numerically.

- Sensitivity:
  - Quantify how a solution to an IEP is subject to changes of the spectral data.

- Applicability:
  - Differentiate whether the given data are exact or approximate, complete or incomplete, and whether only an estimation of the parameters of the system is sufficient.
  - Decide between physical realizability and physical uncertainty which constraint of the problem should be enforced.
• Studies of IEP’s have been quite extensive
  ◇ Engineering application.
  ◇ Algebraic theorization.
• Mathematical techniques employed in the study are quite sophisticated:
  ◇ Algebraic curves.
  ◇ Degree theory.
  ◇ Differential geometry.
  ◇ Matrix theory.
  ◇ Differential equations.
  ◇ Functional analysis.
  ◇ :
• Results are quite few and scattered even within the same field of discipline.
Introduction

Literature Review

• Inverse Sturm-Liouville problem:
  ◦ Ambartsumyan’29
  ◦ Krein’33
  ◦ Borg’46, Levinson’49
  ◦ Gel’fand&Levitan’51
  ◦ Kac’66 (Can one hear the shape of a drum?)
  ◦ Hochstadt’73, Barcilon’74, McLaughlin’76, Hald’78
  ◦ Zayed’82, Issacson et al’83, McLaughlin’86, Andersson’88
  ◦ Lowe et al’95, Rundell’97

• Matrix theory:
  ◦ Downing&Householder’56, Mirsky’58
  ◦ Hochstadt’67
  ◦ de Oliveira’70, Hald’72, Golub’73, Friedland’77, de Boor&Golub’78
  ◦ Biegler-König’81, Shapiro’83, Barcilon’86, Sun’86, Boley&Golub’87
  ◦ Landau’94, Chu’98
Brief History

- Applied Mechanics:
  - Barcilon’74
  - Gottlieb’83, Gladwell’86
  - Ram’91, Gladwell’96, Nylen&Uhlig’97

- Computation:
  - Morel’76, Boley&Golub’77
  - Nocedal et al’83, Friedland et al’88, Laurie’88
  - Chu’90, Zhou&Dai’91, Trench’97, Xu’98
Applications

- System identification and control theory.
  - State/output feedback pole assignment problems.
- Applied mechanics and structure design.
  - Construct a model of a (damped) mass-spring system with prescribed natural frequencies/modes.
- Applied physics.
  - Compute the electronic structure of an atom from measured energy levels.
  - Neutron transport theory.
- Numerical analysis.
  - Preconditioning.
  - Computing $B$-stable RK methods with real poles.
  - Gaussian quadratures.
- Mathematical analysis.
  - Inverse Sturm-Liouville problems.
An Example

- Vibration of equally spaced particles (with spacing $h$ and mass $m_i$) on a string subject to a constant horizontal tension $F$.

- Equation of motion for 4 particles:

\[
\begin{align*}
    m_1 \frac{d^2 x_1}{dt^2} &= -F \frac{x_1}{h} + F \frac{x_2 - x_1}{h} \\
    m_2 \frac{d^2 x_2}{dt^2} &= -F \frac{x_2 - x_1}{h} + F \frac{x_3 - x_2}{h} \\
    m_3 \frac{d^2 x_3}{dt^2} &= -F \frac{x_3 - x_2}{h} + F \frac{x_4 - x_3}{h} \\
    m_4 \frac{d^2 x_4}{dt^2} &= -F \frac{x_4 - x_3}{h} - F \frac{x_4}{h}
\end{align*}
\]
• In matrix form:

\[
\frac{d^2\mathbf{x}}{dt^2} = -DA\mathbf{x}
\]

\[\mathbf{x} = [x_1, x_2, x_3, x_4]^T\]

\[A = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{bmatrix}\]

\[D = \text{diag}(d_1, d_2, d_3, d_4)\] with \(d_i = \frac{F}{m_ih}\).

• Eigenvalues of \(DA\) are the squares of the so called *natural frequencies* of the system.

• Want to place weights \(m_i\) so that the system has a prescribed set of natural frequencies.

  ◦ \(A\) is symmetric and tridiagonal.
  ◦ \(D\) is diagonal.
  ◦ This is a multiplicative inverse eigenvalue problem.

• **Open Question:** Can such a string have arbitrarily prescribed natural frequencies by adjusting the diagonal matrix \(D\)?
Classification

• Based on constraint.
  ◇ Spectral constraint.
  ◇ Structure constraint.
• Based on physical suitability.
  ◇ Physical realizability.
  ◇ Physical uncertainty.
• Based on discipline.
  ◇ Essentially mathematical problem.
  ◇ Essentially engineering problem.
• Based on expectation.
  ◇ Determination problem.
  ◇ Estimation problem.
Via Algebraic Characteristics

MVIEP = Multi-Variate IEP
LSIEP = Least Squares IEP
PIEP = Parameterized IEP
SIEP = Structured IEP
PDIEP = Partially Described IEP
AIEP = Additive IEP
MIEP = Multiplicativc IEP
**PIEP**

- **Generic form:**
  - ◊ **Given**
    - ▶ A *family* of matrices $A(c) \in \mathcal{M}$ with $c \in \mathbb{F}^m$,
    - ▶ A set of scalars $\Omega \subset \mathbb{F}$,
  - ◊ **Find**
    - ▶ Values of parameter $c$ such that
      $$\sigma(A(c)) \subset \Omega$$

- **Remarks:**
  - ◊ Not necessarily $m = n$.
  - ◊ Commonly used $\Omega$:
    - ▶ $\Omega = \{\lambda_*^k\}_{k=1}^n$.
    - ▶ $\Omega = \text{left-half complex plan}$.
    - ▶ $\Omega = \text{anything but must have a specific number of zeros.}$
Some Special PIEP’s

- \( A(c) = A_0 + \sum_{i=1}^{n} c_i A_i \)
  - \( A_i \in \mathcal{R}(n), \ F = \mathbb{R} \).
  - \( A_i \in \mathcal{S}(n), \ F = \mathbb{R} \).

- \((\text{AIEP})\) \( A(c) = A(X) = A_0 + X, \ X \in \mathcal{N} \).
  - \( A_0 \in \mathcal{C}(n), \ F = \mathbb{C}, \ \mathcal{N} = \mathcal{D}_C(n) \).

- \((\text{MIEP})\) \( A(c) = A(X) = XA_0, \ X \in \mathcal{N} \).
  - Preconditioning?

- \( A(c) = A(K_1, \ldots, K_q) = A_0 + \sum_{i=1}^{q} B_i K_i C_i \).
  - Pole assignment problem.
SIEP

- Generic form:
  - Given
    - A set $\mathcal{N}$ of specially structured matrices,
    - A set of scalars $\{\lambda_k^*\}_{k=1}^n \in \mathbf{F}$,
  - Find
    - $X \in \mathcal{N}$ such that
      $$\sigma(X) = \{\lambda_k^*\}_{k=1}^n.$$  

- Some special cases:
  - $\mathcal{N} = \{\text{Toeplitz matrices in } S(n)\}$.
  - $\mathcal{N} = \{\text{Persymmetric Jacobi matrices in } S(n)\}$.
  - $\mathcal{N} = \{\text{Nonnegative matrices in } S(n)\}$.
  - $\mathcal{N} = \{\text{Row-stochastic matrices in } R(n)\}$.
A Few More Special SIEP’s

• Given scalars $\lambda_i^* \leq \mu_i \leq \lambda_{i+1}^*$, $i = 1, \ldots, n - 1$, find a Jacobi matrix $J$ such that

\[
\sigma(J) = \{\lambda_k\}_{k=1}^n, \\
\sigma(J(1:n-1, 1:n-1)) = \{\mu_1, \ldots, \mu_{n-1}\}.
\]

• Given scalars $\{\lambda_1, \ldots, \lambda_{2n}\}$ and $\{\mu_1, \ldots, \mu_{2n-2}\} \subset \mathbb{C}$, find tridiagonal symmetric matrices $C$ and $K$ for the $\lambda$-matrix $Q(\lambda) = \lambda^2 I + \lambda C + K$ so that

\[
\sigma(Q) = \{\lambda_1, \ldots, \lambda_{2n}\}, \\
\sigma(Q(1:n-1, 1:n-1)) = \{\mu_1, \ldots, \mu_{2n-2}\}.
\]

• Given distinct scalars $\{\lambda_1, \ldots, \lambda_{2n}\} \subset \mathbb{R}$ and a Jacobi matrix $J_n \in \mathcal{R}(n)$, find a Jacobi matrix a Jacobi matrix $J_{2n} \in \mathcal{R}(2n)$ so that

\[
\sigma(J_{2n}) = \{\lambda_1, \ldots, \lambda_{2n}\}, \\
J_{2n}(1:n, 1:n) = J_n.
\]

• Given a family of matrices $A(c) \in \mathbb{R}^{m \times n}$, with $c \in \mathbb{R}^n$, $m \geq n$, find a parameter $c$ such that the singular values of $A(c)$ are precisely the same as a prescribed set of nonnegative real values $\{\sigma_1, \ldots, \sigma_n\}$.
LSIEP

- Maintain the structure, approximate the eigenvalues:
  - Given
    - A set of scalars \( \{\lambda_1^*, \ldots, \lambda_m^*\} \subset \mathbb{F} \ (m \leq n) \),
    - A set \( \mathcal{N} \) of specially structured matrices,
  - Find
    - A matrix \( X \in \mathcal{N} \)
    - An index subset \( \sigma = \{\sigma_1 < \ldots < \sigma_m\} \) such that
      \[
      F(X, \sigma) := \frac{1}{2} \sum_{i=1}^{m} (\lambda_{\sigma_i}(X) - \lambda_i^*)^2,
      \]
      is minimized.
• Maintain the spectrum, approximate the structure:

  ◆ Given
  ▶ A set $\mathcal{M}$ of spectrally constrained matrices,
  ▶ A set $\mathcal{N}$ of specially structured matrices,
  ▶ A projection $P$ from $\mathcal{M}$ onto $\mathcal{N}$,

  ◆ Find
  ▶ $X \in \mathcal{M}$ that minimizes
  \[
  F(X) := \frac{1}{2} \| X - P(X) \|^2.
  \]
Via Physical Characteristics

- By mechanical types:
  - Continuous vs. discrete.
  - Damped vs. undamped.
- By data type:
  - Spectral, modal, or nodal.
  - Complete vs incomplete.
A Glimpse of Some Major Issues

- Studies on IEP’s have been intensive, ranging from acquiring a pragmatic solution to a real-world application dealing the metaphysical theory of an abstract formulation.

- Results are scattered even within the same field of discipline.

- Only a handful of the problems have been completely understood.

- Many interesting yet challenging questions remain to be answered.
Complex Solvability

- Solving an IEP over complex field amounts to solving a polynomial system with complex coefficients. Generally speaking, the system is generically solvable.

- Given $A_0 \in \mathcal{C}(n)$ and arbitrary $\{\lambda_k^*\}_{k=1}^n \subset \mathbb{C}$,
  - There exists $D \in \mathcal{D}_C(n)$ such that
    \[
    \sigma(A_0 + D) = \{\lambda_k^*\}_{k=1}^n
    \]
    and there are at most $n!$ solutions.
  - If $\det(A_0(1:j, 1:j)) \neq 0$, $j = 1, \ldots, n$, then there exists $D \in \mathcal{D}_C(n)$ such that
    \[
    \sigma(DA_0) = \{\lambda_k^*\}_{k=1}^n
    \]
    and there are at most $n!$ solutions.
Real Solvability

- Solving an IEP over real field is a much harder problem. Sufficient conditions are generally quite restrictive.
- Assume all matrices involved are real,
  - If the prescribed real eigenvalues are sufficiently different, then there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that
    \[
    \sigma(A_0 + \sum_{i=1}^{n} c_i A_i) = \{\lambda_k^*\}_{k=1}^{n}.
    \]
  - The inverse eigenvalue problem associated with
    \[
    A_0 + \sum_{i=1}^{n} c_i A_i
    \]
    is unsolvable almost everywhere if and only if any of the prescribed eigenvalues has multiplicity greater than 1.
- Symmetric Toeplitz matrices can have arbitrary spectra.
Numerical Methods

- Direct methods
  - Lanczos method.
  - Orthogonal reduction methods.
- Iterative methods
  - Newton-type iteration.
- Continuous methods:
  - Homotopy approach.
  - Projected gradient method.
  - ASVD approach.
Sensitivity Analysis

- Assume all matrices are symmetric and the PIEP for

\[ A(c) = A_0 + \sum_{i=1}^{n} c_i A_i \]

is solvable.

- Assume \( A(c) = Q(c) \text{diag}\{\lambda_k^*\}_{k=1}^{n} Q(c)^T \) and define

\[ J(c) = [q_i(c)^T A_j q_i(c)], \quad i, j = 1, \ldots, n, \]

\[ b = [q_1(c)^T A_0 q_1(c), \ldots, q_n^T A_0 q_n(c)]^T. \]

- If

\[ \delta = \|\lambda - \tilde{\lambda}\|_{\infty} + \sum_{i=0}^{n} \|A_i - \tilde{A}_i\|_2 \]

is sufficiently small, then

\[ \diamond \text{The PIEP associated with } \tilde{A}_i, \ i = 0, \ldots, n \text{ and } \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n\} \text{ is solvable.} \]

\[ \diamond \text{There is a solution } \tilde{c} \text{ near to } c, \]

\[
\frac{\|c - \tilde{c}\|_{\infty}}{\|c\|_{\infty}} \leq \kappa_{\infty}(J(c)) \left( \frac{\|\lambda - \tilde{\lambda}\|_{\infty} + \|A_0 - \tilde{A}_0\|_2}{\|\lambda - b\|_{\infty}} + \frac{\sum_{i=1}^{n} \|A_i - \tilde{A}_i\|_2}{\|J(c)\|_{\infty}} \right) + O(\delta^2). \]
Summary

- An IEP concerns the reconstruction of a matrix satisfying two constraints.
  - *Spectral constraint* – the prescribed spectral data.
  - *Structural constraint* – the desirable structure.
- Different constraints define a variety of IEP’s.
- Studies on IEP’s have been intensive, ranging from engineering application to algebraic theorization.
  - Many unanswered yet interesting questions.
- A common phenomenon in all applications is that the physical parameters of a certain system are to be reconstructed from knowledge of its dynamical behavior, in particular, of its natural frequencies/modes.
  - Sometimes the constraints can be precisely determined.
  - Sometimes the constraints are only approximate and often incomplete.