1: Rules for differentiation

- Power function: \( h = f \Rightarrow h' = f' \)
- Exponential function: \( h = e^x \Rightarrow h' = e^x \)
- Logarithmic function: \( h = \log(x) \Rightarrow h' = \frac{1}{x} \)
- Trigonometric functions: \( \sin(x) \), \( \cos(x) \)

**Product rule**: \( h = f \cdot g \Rightarrow h' = f' \cdot g + f \cdot g' \)

**Quotient rule**: \( h = \frac{f}{g} \Rightarrow h' = \frac{g f' - f g'}{g^2} \)

**Chain rule**: \( h = f(g(x)) \Rightarrow h' = f'(g(x)) \cdot g'(x) \)

**Examples**

- \( f(x) = (4x^2+3)(1/x - 4) \)
- \( f(x) = (\sin(x)+3x) / (\cos(2x) - 4) \)
- \( f(x) = \exp(x^3 + 2x-4) \)

2: Limits

- \( \lim_{x \to \infty} \frac{1}{x} = 0 \)
- \( \lim_{x \to 0} \frac{1}{x} = \text{DNE} \)
- \( \lim_{x \to \infty} x = \text{DNE} \)
- \( \lim_{x \to 0} x = 0 \)
- \( \lim_{x \to \infty} \sin(x) / \cos(x) \) oscillate between -1 and 1
- \( \lim_{x \to \infty} \sqrt{x} = \text{DNE} \)
- \( \lim_{x \to \infty} \exp(x) \) and \( \ln(x) = \text{DNE} \)

**Examples**

- \( \lim_{x \to \infty} f(x) = \frac{4x^3 - 2x^2 - 3x + 2}{2x^3 + x - 3} = 4/2 \)
- \( \lim_{x \to \infty} f(x) = \frac{x^3 + 4x}{3x^2 + 2x} = \text{DNE} \)
- \( \lim_{x \to 0} f(x) = \frac{x^3 + 4x}{3x^2 + 2x} = 0 \)
- \( \lim_{x \to \infty} f(x) = \frac{x^2 + 3x - 2}{2x^3 - x^2 + 2} = 0 \)

3: Distance, velocity, acceleration

- \( s(t) = \text{distance (distance miles / feet / in / meter / cm)} \)
- \( v(t) = ds/dt = \text{velocity = distance / time (time = months / days / hours / min / sec)} \)
- \( a(t) = dv/dt = \text{acceleration = velocity / time} \)

\( v(t) = \int (a(t)) \) up to constant
\( d(t) = \int (v(t)) \) up to constant

**Example**:

The velocity of an object thrown up in the air is given by \( v(t) = -4t + 10 \text{ m/s} \). At time 0, the height of the object is 5.5 m.
Find the acceleration of the object?

\[ a(t) = \frac{dv}{dt} = -4 \text{ (constant)} \]

What is the acceleration at \( t = 2 \) sec?

\( -4 \) it is always \(-4\) for any \( t \) (constant acceleration)

When is the velocity \( v = 0 \) meter/sec?

When \( v(t) = 0 \) i.e. \( 0 = -4t + 10 \iff 4t = 10 \iff t = 10/4 \).

What is the max height the object reaches?

When \( v(t) = 0 \) (i.e. when \( s'(t) = 0 \)) so at \( t = 10/4 \) as we found above

What is the average velocity between \( t = 3 \) and \( t = 5 \) sec?

Secant line: \((s(5) - s(3))/2 = (5.5 - 17.5)/2 = -6\)

Compare with the velocity at \( t = 4 \)

\( v(4) = -6 \)

Find the distance the object travels?

\[ s(t) = \int (v(t)) + c = \int -4t + 10 = -2t^2 + 10t + c \]

\[ s(0) = 5.5 \Rightarrow c = 5.5. \]

\[ s(t) = \int (v(t)) + c = \int -4t + 10 = -2t^2 + 10t + 5.5 \]

When does the object hit the ground?

When \( s(t) = 0 \) i.e. \( 0 = -2t^2 + 10t + 5 \iff \)

\[ t = (-10 +/- \sqrt{100 + 4 * 2 * 5.5})/-4 = (-10 +/- 12)/(-4) \]

Only positive solution makes sense so \( t = (-10 - 12)/(-4) = 22/4 = 5.5 \) h

Sketch the graph for \( s(t) \)

Max (\( t = 10/4 \))

Concavity (concave down, \( s'' = a(t) = -4 \) negative for all \( t \))

x-intercept \( s(t) = 0 \) t = -.5 (not in domain) and t = 5.5

y-intercept \( t = 0 \Rightarrow s(t) = 5.5 \)

In general you also need to check inflection points \( f'' = 0 \) (point at which the function changes concavity)

**4: Optimization**

A) Find objective function (what should be optimized)

B) Find constraint function (an equation giving a number)

C) Use constraint function to simplify the objective function

D) Minimize/Maximize

E) Check that it is a minimum (2\(^{nd}\) order rule)

F) Predict properties
Example:
Norman wanted to paint a picture. The picture should be hung 1 foot from the ceiling, 2 feet from the floor, and 4 feet from either side. The wall has an area of 160 sqr feet.

Find the objective function (the picture area)
Ap = w * h

Find the constraint function?
A = 160 = w * h
wp + 8 = 160 / (hp + 3) \Leftrightarrow wp = 160 / (hp + 3) - 8

Simplify the objective function
Ap = (160/(hp + 3) - 8) * hp
    = (160 hp / (hp + 3) - 8 hp

Maximize the objective function
Ap' = 0
Ap' = ((hp + 3) 160 - 160 hp 1) / (hp + 3)^2 - 8
Ap' = (160 hp + 480 - 160 hp) / (hp + 3)^2 - 8
Ap' = (480) / (hp + 3)^2 - 8 = 0

8 = (480) / (hp + 3)^2
8 ( hp^2 + 9 + 6 hp ) = 480
8 hp^2 + 72 + 48 hp = 480
8 hp^2 + 48 hp - 408 = 0
hp = -48 +/- sqrt (2304 + 13056) / 16
hp = (-48 + 123.9355) / 16 = 4.7460 [ - solution makes no sense ]

Find all dimensions (height and width)
hp = 4.746
wp = 160 / (hp + 3) - 8 = 160 / (4.746 + 3) - 8 = 12.6558

Check that the point is a maximum
Ap' = (480) / (hp + 3)^2 - 8 = 0
Ap'' = -480 / (hp + 3)^4 < 0 \Leftrightarrow concave down \Rightarrow maximum

5: Exponential growth/decay / half time / doubling time
The solution to the equation
P'(t) = k * p(t)
is given by:
p(t) = p_0 exp(kt)
    k > 0 exponential growth
    k < 0 exponential decay
Half time at which \( p(t) = \frac{p_0}{2} \)
Doubling time at which \( p(t) = 2p_0 \)

**Examples:**
If it takes 6 hours to double the number of bacteria in a dish, how long will it take before there are 100 times as many bacteria?

\[
2p_0 = p_0 \exp(kt) \quad \Rightarrow \quad k = \frac{\log(2)}{6} = 0.1155
\]
\[
100p_0 = p_0 \exp(0.1155t) \quad \Rightarrow \quad t = \frac{\log(100)}{0.1155} = 39.8716
\]

If the half time for a given radioactive isotope is 3 hours, how long would it take to reduce the original amount of isotope to 1/10?

\[
p_0/2 = p_0 \exp(3k) \quad \Rightarrow \quad k = \frac{\log(1/2)}{3} = -0.2310
\]
\[
p_0/10 = p_0 \exp(-0.2310t) \quad \Rightarrow \quad t = \frac{\log(1/10)}{-0.2310} = 9.9679
\]

You find a skull in a nearby Native American ancient burial site and with the help of a spectrometer, discover that the skull contains 9% of the C-14 found in a modern skull.

Assuming that the half-life of C-14 is 5730 years, how old is the skull?

At the start of an experiment, there are 100 bacteria. If the bacteria follow an exponential growth pattern with rate \( k = 0.02 \), what will be the population after 5 hours? How long will it take for the population to double?

Answer: 34:6574 hours

Suppose that the population of a colony of bacteria increases exponentially. At the start of an experiment, there are 6,000 bacteria, and one hour later, the population has increased to 6,400. How long will it take for the population to reach 10,000?

Round your answer to the nearest hour.

Answer: 8 hours.

Use the fact that the world population was 2560 million people in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the growth rate \( k \)? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Answer: \( k = 0.017185 \), 1993 pop = 5360 mill, 2020 pop = 8524 mill