BMA 771
HW 2

For 2.1.1-2.1.3 \( \dot{x} = \sin x \)

2.1.1. Find all fix points for the flow
The fixed points occur where \( \dot{x} = 0 \), or \( \sin x = 0 \). This occurs when \( x = n\pi, n \in \mathbb{Z} \), so all of the fixed points of \( x \) are of the form above. Fix-points for \( n \) odd are stable.

2.1.2. At which points does the flow have the greatest velocity to the right?
Since \( \dot{x} \) is the sine function, it is common knowledge that the local maxima occur at \( x = \frac{\pi}{2} + 2\pi n, n \in \mathbb{Z} \), which is where the sine function equals 1. \( \dot{x} \) represents the velocity of the flow, so the local maxima defined above are where the velocity is the greatest to the right.

2.1.3. Find the flow’s acceleration as a function of \( x \)
a) \[
\ddot{x} = \frac{d(\dot{x})}{dt} = \frac{d(\sin(x))}{dt} = \cos(x) \dot{x} = \cos(x) \sin(x) = \sin(2x) / 2
\]
b) The local maxima for \( \dot{x} \) occur when \( x = \frac{\pi}{4} + \pi n, n \in \mathbb{Z} \), or where \( \dot{x} = 1/2 \). These points, represent where the flow has the maximum positive acceleration since \( \dot{x} \) is the formula for the acceleration of the flow in terms of \( x \).

2.1.4. Show that the solution for the equation can be converted to find \( x(t) \)
a) \[
t = \ln \frac{\csc(\frac{\pi}{4}) + \cot(\frac{\pi}{4})}{\csc(x) + \cot(x)} \Rightarrow
e^{t} = \frac{\sqrt{2} + 1}{\csc(x) + \cot(x)} = \frac{\sqrt{2} + 1}{\frac{\sin(x)}{\cos(x)} + \frac{\cos(x)}{\sin(x)}} = \frac{\sqrt{2} + 1}{\sin(x)} = \frac{\sin(x)}{1 + \cos(x)}
e^{t} = \frac{\sin(x)}{1 + \cos(x)} = \frac{2 \sin(\frac{x}{2}) \cos(\frac{x}{2})}{1 + 2 \cos^{2}(\frac{x}{2}) - 1} = \tan(\frac{x}{2}) \rightarrow x(t) = 2 \tan^{-1}\left(\frac{e^{t}}{\sqrt{2} + 1}\right)
\]

In the above equality, as \( t \to \infty, x(t) \to \pi \) because \( \pi/2 \) is a vertical asymptote for \( \tan(x) \), and \( \tan(x) \to \infty \) as \( x \to \pi \).
b) Try to find the analytical solution for \( x(t) \) given an arbitrary initial condition \( x_0 \)

The analytical solution for \( x(t) \) when \( x_0 \) is arbitrary instead of \( \pi/4 \) is:

\[
x(t) = 2 \tan^{-1} \left( \frac{e^t}{\csc(x_0) + \cot(x_0)} \right)
\]

This is directly interpreted from the calculations done in part a). This can be somewhat simplified to \( x(t) = 2 \tan^{-1} \left( \frac{e^t}{\cot(x_0/2)} \right) \) using double angle formulas.

2.1.5.

a) Find a mechanical system that is approximately governed by \( \dot{x} = \sin(x) \)

A simple pendulum.

b) Using physical intuition, explain why \( x^* = 0 \) is unstable and \( x^* = \pi \) is stable.

At \( x^* = 0 \) the pendulum is not moving. Any small perturbation away from this fix point will cause the pendulum to swing. At \( x^* = \pi \), however, any small perturbation away from this point will be attracted to the fix point.

2.2.3. Analyze the equation graphically. \( \dot{x} = x - x^3 \). The fixed points are \( x = -1, 0, 1 \). Judging from the Figure 2.3.1 below, the fix points \( x = -1, 1 \) are stable and \( x = 0 \) is unstable. A sketch of \( x(t) \) for some initial conditions is given in Figure 2.3.2. A solution to this differential equation can be obtained using separation of variables.

![Figure 2.3.1](image1)

![Figure 2.3.2](image2)
To solve \( \dot{x} = x - x^3 \) analytically I start with separation of variables.

\[
\int \frac{dx}{x(1-x)(1+x)} = \int dt
\]

By partial fraction decomposition we can get the integral of the RHS to become

\[
\int \frac{1}{x} + \int \frac{1}{2(1-x)} - \int \frac{1}{2(1+x)}
\]

Solving the integral and simplifying results in

\[
\ln \left( \frac{x}{\sqrt{1-x^2}} \right) = t + C
\]

\[
\frac{x}{\sqrt{1-x^2}} = C e^t
\]

\[
x^2 = C e^{2t}
\]

\[
x^2 + C e^{2t} e^{2t} = C e^{2t}
\]

\[
x^2 = \frac{C e^{2t}}{1 + C e^{2t}}
\]

Therefore, the analytical solution to the system \( \dot{x} = x - x^3 \) is

\[
x(t) = \sqrt{\frac{C e^{2t}}{1 + C e^{2t}}}.
\]

**Question 2.2.5**

The system \( \dot{x} = 1 + \frac{1}{2} \cos(x) \) does not have any fixed points. Analytically this can be seen

\[
\frac{1}{2} \cos(x) = -1
\]

\[
\cos(x) = -2
\]

However, the function \( \cos(x) \neq -2 \). By graphing \( x \) vs \( \dot{x} \) it can also be seen that the function never crosses the x-axis and therefore does not have any fixed points.
Analytical solution

\[
\frac{dx}{dt} = \frac{2 + \cos(x)}{2}
\]

\[
\frac{dx}{2 + \cos(x)} = \frac{1}{2} \, dt
\]

\[
\frac{dx}{1 + 1 + \cos(x)} = \frac{1}{2} \, dt
\]

Wolfram Alpha can find the solution giving

\[
x(t) = 2 \tan^{-1}\left(\sqrt{3} \tan\left(\frac{\sqrt{3} \, c_1 + \sqrt{3} \, t}{4}\right)\right)
\]
2.3

Population growth examples

2.3.3 (Tumor growth) Growth of cancerous tumors can be modeled by the Gompertz law
\[ N = -aN \ln(bN), \]
where \( N(t) \) is proportional to the number of cells in the tumor, and \( a, b > 0 \),
are parameters.
a.) Interpret \( a \) and \( b \) biologically

\('a' is the intrinsic growth rate of the cell, and \('b^{-1}' is the environmental carrying capacity.\)

b.) Sketch the vector field and graph, \( N(t) \), for various initial values.
The fixed points for \( N^* \) are
\[
\begin{align*}
-aN \ln(bN) &= 0 \\
\ln(bN) &= 0 \\
bN(t) &= e^0 \\
bN(t) &= 1 \\
N^*(t) &= 0, \frac{1}{b}
\end{align*}
\]

With the following vector fields for varying initial conditions and parameter values

Figure 3: Vector field 2.3.3.1

Figure 4: Vector field 2.3.3.2
1. Show that $\dot{N}/N = r - a(N - b)^2$ provides an example of Allee effect, if $r, a,$ and $b$ satisfy certain constraints, to be determined.

Assuming $r, a, b > 0$, we found that this equation does create a weak and a strong Allee effect. The weak Allee effect is characterized by very slow growth at low populations due to lack of mates. In contrast, the strong Allee effect is characterized by death at low populations due to the lack of mates. To show the existence of both types of Allee effects, we used the Manipulate command in Mathematica to find certain values of $a$, $b$, and $r$ that create an Allee effect (code attached at end). We used the fixed points (see below) to differentiate between the weak and the strong Allee effect. The weak effect occurs when $r > ab^2$ and the strong effect occurs when $r < ab^2$.

2. Find all the fixed points of the system and classify their stability.

To find the fixed points, we solve for $\dot{N} = f(N) = 0$.

$$Nr - aN(N - b)^2 = N(r -aN^2 - 2bN + b^2) = N(-aN^2 + 2abN + r - ab^2) = 0$$

Therefore, the three fixed points are

$$N^* = 0, N^* = \frac{-2ab \pm \sqrt{4a^2b^2 + 4a(r - ab^2)}}{-2a} = b \pm \frac{\sqrt{r}}{\sqrt{a}}$$

To classify their stability, we use linear stability analysis by looking at $f'(N^*)$ for each $N^*$. First note that $f'(N) = r - 3aN^2 + 4abN - ab^2$.

$$N^* = 0 : f'(0) = r - ab^2. \text{ If } r > ab^2, f'(0) > 0 \text{ and } N^* = 0 \text{ is a unstable fixed point. If } r < ab^2, f'(0) < 0 \text{ and } N^* = 0 \text{ is a stable fixed point.}$$

$$N^* = b + \frac{\sqrt{r}}{\sqrt{a}} : f'(b + \frac{\sqrt{r}}{\sqrt{a}}) = r - 3a(b + \frac{\sqrt{r}}{\sqrt{a}})^2 + 4ab(b + \frac{\sqrt{r}}{\sqrt{a}}) - ab^2 = -2r - 2b\sqrt{a}\sqrt{r}.$$

Since $a, b, r > 0$, then clearly $-2r - 2\sqrt{a}\sqrt{r}b < 0$. Therefore, $N^* = b + \frac{\sqrt{r}}{\sqrt{a}}$ is a stable fixed point.

$$N^* = b - \frac{\sqrt{r}}{\sqrt{a}} : f'(b - \frac{\sqrt{r}}{\sqrt{a}}) = r - 3a(b - \frac{\sqrt{r}}{\sqrt{a}})^2 + 4ab(b - \frac{\sqrt{r}}{\sqrt{a}}) - ab^2 = 2b\sqrt{a}\sqrt{r} - 2r = b - \frac{\sqrt{r}}{\sqrt{a}}. \text{ If } b > \frac{\sqrt{r}}{\sqrt{a}}, \text{ the fixed point is unstable.}$$

3. Sketch the solutions $N(t)$ for different initial conditions.

We were not able to find an analytical solution to the equation using separation of variables or change of variables. Attempts to solve the equation using Matlab and Mathematica's DSolve function were also unsuccessful. Therefore, the numerical solutions found using the second order Runge-Kutta method and the ODE45 Matlab function were plotted in Figure (9) and Figure (10) using the initial conditions $N(0) = 1$ and $N(0) = 7$ and the parameters for the strong Allee effect found in Mathematica. Note
According to the graphical analysis on fig. 4a, for the case with 3 fixed points, \( N = 0 \) and \( N = b + \sqrt{r/a} \) are stable, and \( N = b - \sqrt{r/a} \) is unstable. Fig. 5a shows that in case of 2 fixed points \( N = 0 \) is unstable and \( N = b + \sqrt{r/a} \) is stable.

\[
\frac{N'}{N} = r - a (N-b)^2
\]

Figure 4: Graphical analysis of \( \dot{N} = N (r - a(N - b)^2) \). 3 fixed points case.

\[
\frac{N'}{N} = r - a (N-b)^2
\]

Figure 5: Graphical analysis of \( \dot{N} = N (r - a(N - b)^2) \). 2 fixed points case.
2.4

Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x^*) = 0$, then use the graphical argument to decide stability.

2.4.2 $\dot{x} = x(1 - x)(2 - x)$

Taking $\dot{x} = 0$, we have

$$x(1 - x)(2 - x) = 0$$

$$x^*(t) = 0, 1, 2$$

and taking

$$f(x) = 2x - 3x^2 + x^3$$
$$f'(x) = 3x^2 - 6x + 2$$

and evaluating the derivative of $f$ at the fixed points we have

$$f'(0) = 2 \Rightarrow \text{unstable}$$
$$f'(1) = -1 \Rightarrow \text{stable}$$
$$f'(2) = 2 \Rightarrow \text{unstable}$$

2.4.8 Using linear stability analysis, classify the fixed points of the Gompertz model of tumor growth, $\dot{N} = -aN \ln(bN)$.

From 2.3.3 we have our fixed points

$$N^*(t) = 0, \frac{1}{b}.$$  

Defining

$$f(N) = -aN \ln(bN)$$

such that

$$f'(N) = -a \ln(bN) - a.$$  

Evaluating the derivative of $f$ at the fixed points we have

$$f'(0) = -a \ln(0) - a$$

$\rightarrow \infty$
Implying that our fixed point $N^* = 0$ is unstable $\forall a > 0$. Taking

$$f'(\frac{1}{b}) = -a \ln(b \cdot \frac{1}{b}) - a$$
$$= -a \ln(1) - a$$
$$= -a$$

Showing that our fixed point $N^* = \frac{1}{b}$ is stable $\forall a > 0$.

**2.5**

2.5.1 (Reaching a fixed point in a finite time) A particle travels on the half-line, $x \geq 0$, with a velocity given by $\dot{x} = -x^c$, where $c \in \mathbb{R}$ is a constant.

a.) Find all values of $c$ such that the origin, $x = 0$, is a stable fixed point

Letting $x^* = 0$ to be our fixed point, graphically we can see that it is stable $\forall c > 0$

![Figure 7: Vector field 2.3.3.2](image)

b.) Now assume that $c$ is chosen such that $x = 0$ is stable. Can the particle ever reach the origin in a finite time? If so, how long does it take for the particle to travel from $x = 0$ to $x = 1$, as a function of $c$?

$$\frac{dx}{dt} = -x^c$$

$$\int \frac{dx}{-x^c} = \int dt$$

$$\int -x^{-c}dx = t + k$$

$$\frac{-x^{1-c}}{1-c} = t + k$$

$$\frac{x^{1-c}}{c-1} = t + k$$

For $x = 0$, we have $t_0 = -k$, and for $x = 1$, we have $t_1 = \frac{1}{c-1} - k$.

This gives us

$$t_1 - t_0 = \frac{1}{c-1}$$

2.5.3 Consider the equation $\dot{x} = rx + x^3$ where $r > 0$ is fixed. Show that $x(t) \to \pm\infty$ in finite
time, starting from any initial condition \( x_0 \neq 0 \).

Taking,

\[
\frac{dx}{dt} = rx + x^3
\]

\[
\int \frac{dx}{x(r + x^2)} = \int dt
\]

\[
\int \frac{dx}{rx} - \int \frac{xdx}{r(r + x^2)} = t + c
\]

\[
\ln(x) \sqrt{r(r + x^2)} = t + c
\]

As \( x \to \infty \), \( \ln \left( \frac{x}{r + x^2} \right) \to 0 \). Then we can say \( t = k \) for some constant, \( k \), for any \( x_0 \neq 0 \).

2.6

2.6.1 Explain the paradox: a simple harmonic oscillator \( m\ddot{x} = -kx \) is a system that oscillates in one dimension (along the x-axis). But the text says one-dimensional systems can’t oscillate.

\( m\ddot{x} = -kx \) is a second-order ODE; which can be expressed as a system of two first-order ODE’s, therefore making the oscillations possible.

2.7

For the following vector fields, plot the potential function \( V(x) \) and identify all equilibrium points and their stability

2.7.3 \( \dot{x} = \sin(x) \)

Let \( f(x) = \sin(x) \)

\[
\frac{dV}{dx} = f(x)
\]

\[
\frac{dV}{dx} = \sin(x)
\]

\[
\int dV = -\int \sin(x)dx
\]

\( V(x) = \cos(x) + c \)

Taking the constant, \( c = 0 \), and

\( \cos(x) = 0 \)

\( x^+(t) = \cos^{-1}(0) \)

\( x^+(t) = \pm \pi , \forall n \in \mathbb{Z} \)

Graphing \( V(x) \) we have
Observing the local minima occur where \( n \) is odd, and the local maxima occur where \( n \) is even, we have

For \( n = 2k + 1, \ k \in \mathbb{Z} \Rightarrow \text{stable} \)

For \( n = 2k, \ k \in \mathbb{Z} \Rightarrow \text{unstable} \)

**Numerical Methods**

Solve 2.4.8 equation using Euler's method, the 2-step Runge Kutta method (2 methods with \( w_1 = w_2 = 1/2 \) and \( \alpha = \beta = 1 \) and using \( w_1 = 0, w_2 = 1, \alpha = \beta = 1/2 \)), and using Matlab's build in ODE solver ode45. Calculate the error between analytical and numerical solutions. What do you observe?

For the differential equation we found our analytic solution to be

\[
N(t) = N_0 e^{\frac{b}{a} (1 - e^{-at})}.
\]

Using the parameter and initial condition values from the literature used, we found the plots for the errors between the numerical and analytic solutions seemed to behave like a logistic function leveling off around 5. With the parameter estimations for \( N_0 = 0, a = 0.159, \) and \( b = 1.02 \), we were returned the following plots:

![Figure 9: Plot w/ analytic](image-url)
Figure 10: Error Plot

Examining the errors we found that they seem relatively high, but from the literature the cellular carrying capacity for the environment regarding the growth of Osteosarcomas was in the range of $10^5$. With respect to the magnitude of the carrying capacity, the errors are relatively small [1].

Set parameter values and initial values such that the model equations represent real biological populations. Include a paragraph (with a literature reference) describing the populations chosen.

For parameter values for the Gompertz tumor growth problem, we used a paper that worked on uniqueness and sensitivity analysis of parameter values for the Gompertz tumor growth model. For one of the analyses they used Osteosarcomas, and they determined that $a = 0.159$, $b = 1.02$, with a Volume Rate Doubling time (VRD) of $VRD = 0.7191$[1].
MATLAB

function eulerexample2

tstart = 0; % Start time

tend = 5; % End time

dt = 0.1; % Delta t

t = [0:dt:tend];

N = length(t);

y0 = 0.1; % Initial condition

yeu(1) = y0; % Apply the initial condition
yan(1) = y0;
yrk1(1) = y0;
yrk2(1) = y0;
% The general ODE: dy/dt = f(t,y)

% Our example autonomous ODE: dy/dt = f(0,y) = y(1-y)
% the small step size h = dt

%Eulers method: y_n+1 = y_n + dt f(t_n, y_n)

for i = 2:N
    yeu(i) = yeu(i-1) + fct(yeu(i-1))*dt;
end;

%2nd order RK method #1: w1=w2=1/2; alpha=beta=1

for i = 2:N
    K1 = dt * fct(yrk1(i-1));
    K2 = dt* fct (yrk1(i-1) + K1);
    yrk1(i) = yrk1(i-1) + 1/2*K1 + 1/2*K2;
end;

%2nd order RK method #2: w1=0; w2=1; alpha=beta=1/2

for i = 2:N
    L1 = dt * fct(yrk2(i-1));
    L2 = dt* fct (yrk2(i-1) + 1/2*L1);
    yrk2(i) = yrk2(i-1) + 1/2*L2;
end;

% Solved numerically using matlab

options = odeset('RelTol',1e-6,'Abstol',1e-6);
sol = ode45(@rhs, [tstart tend], y0, options); % function name, time interval, initial conditions, options
yod = deval(sol, t); % interpolates the solution at t = 0 0.1 0.2 0.3 ....

% Analytical solution:

yan = .01*exp((1.02/0.159)*exp(1-1.02*t));

% Error

Eeu = abs(yeu-yan)
Erk1 = abs(erk1-yan)
Erk2 = abs(erk2-yan)
Eod = abs(yod-yan)

figure(3);
h=plot(t, y, 'k', t, yeu, 'b', t, yrk1, 'g', t, yrk2, 'r', t, yod, 'c');
set(h,'linewidth',2);
set(gca,'fontsize',20);
xlabel('time');
ylabel('y');
legend('ana', 'eul', 'rk1', 'rk2', 'ode45');
xlim([0 5]);
grid;

figure(6);
h=plot(t, y, 'k', t, yeu, 'b', t, yrk1, 'g', t, yrk2, 'r', t, yod, 'c');
set(h,'linewidth',2);
set(gca,'fontsize',20);
xlabel('time');
ylabel('y');
legend('ana', 'eul', 'rk1', 'rk2', 'ode45');
xlim([0 5]);
grid;

figure(4);
h=plot(t, Eeu, t, Erk1, t, Erk2, 'r', t, Eod);
set(h,'linewidth',2);
set(gca,'fontsize',20);
xlabel('time');
ylabel('error');
legend('euler', 'rk1', 'rk2', 'ode45');
xlim([0 5]);
grid;

figure(5);
h=plot(t, Erk1, t, Eod, 'r');
set(h,'linewidth',2);
set(gca,'fontsize',20);
xlabel('time');
ylabel('error');
xlim([0 5]);
grid;

figure(6);
h=plot(t,Eod,'r');
set(h,'linewidth',2);
set(gca,'fontsize',20);
xlabel('time');
ylabel('error');
xlim([0 5]);
grid;

function dy = rhs(t,y)

dy = -0.159*y*log(1.02*y);

function f = fct(y)

f = -0.159*y*log(1.02*y);
References