somewhere (at an initial point) and has an orientation consistent with increasing values of time. We shall presently see that these ideas have a natural and important generalization to systems of differential equations.

5.2 SYSTEMS OF TWO FIRST-ORDER ODEs

In modeling biological systems, which are generally composed of several interacting variables, we are frequently confronted with systems of nonlinear ODEs. The ideas of Section 5.1 can be extended to encompass such systems; in the present section we deal in great detail with systems of two equations that describe the interaction of two species. The reason for dealing almost exclusively with these will emerge after some preliminary familiarity is established.

Let us therefore turn attention to a system of two autonomous first-order equations, a prototype of which follows:

\[
\frac{dx}{dt} = f_1(x, y), \\
\frac{dy}{dt} = f_2(x, y).
\]  

(5a)  

(5b)

Technically, we assume that \( f_1 \) and \( f_2 \) are continuous functions having partial derivatives with respect to \( x \) and \( y \); this ensures existence of a unique solution given an initial value for \( x \) and \( y \). A solution to system (5) would be two functions, \( x(t) \), and \( y(t) \), that satisfy the equations together with the initial conditions, if any.

As a preliminary to understanding the equations, let us consider an approximate form of these equations, whereby derivatives are replaced by finite differences, as follows:

\[
\frac{\Delta x}{\Delta t} = f_1(x, y), \\
\frac{\Delta y}{\Delta t} = f_2(x, y).
\]  

(6a)  

(6b)

The changes \( \Delta x \) and \( \Delta y \) in the two independent variables are thus specified whenever \( x \) and \( y \) are known, since
\[ \Delta x = f_1(x, y) \Delta t, \quad \Delta y = f_2(x, y) \Delta t. \]  

These equations can be interpreted as follows: Given a value of \( x \) and \( y \), after some small increment of time \( \Delta t \), \( x \) will change by an amount \( \Delta x \) and \( y \) by an amount \( \Delta y \). This is represented pictorially in Figure 5.5, where a point \((x, y)\) is assigned a vector with components \((\Delta x, \Delta y)\) that describe changes in the two variables simultaneously. We see that equations (6) and (7) are mathematical statements that assign a vector (representing a change) to every pair of values \((x, y)\).

\[ \begin{array}{c}
\text{(a)} \\
\text{(b)} \\
\end{array} \]

*Figure 5.5 (a) Given a point \((x, y)\), (b) a change in its location can be represented by a vector \(v\).*

In calculus such concepts are made more precise. Indeed, we know that derivatives are just limits of expressions such as \(\Delta x/\Delta t\) when ever-smaller time increments are considered. Using calculus, we can understand equations (5a,b) directly without resorting to their approximated version. (A review of these ideas is presented in Section 5.3, which may be skipped if desired.)

### 5.3 CURVES IN THE PLANE

In calculus we learn that the concepts point and vector are essentially interchangeable. The pair of numbers \((x, y)\) can be thought of as a point in the cartesian plane with coordinates \(x\) and \(y\) [as in Figure 5.6(a)] or as an arrow strung out between the origin \((0, 0)\) and \((x, y)\) that points to the location of this point [Figure 5.6(b)]. When the coordinates \(x\) and \(y\) vary with time or with some other parameter, the point \((x, y)\) moves over the plane tracing a curve as it moves. Equivalently, the arrow twirls and stretches as its head tracks the position of the point \((x(t), y(t))\). For this reason, it is often called a position vector, symbolized by \(x(t)\).
As previously remarked, since the solution of a system of equations such as 
(5a,b) is a pair \((x(t), y(t))\), the idea that a solution corresponds geometrically to a curve carries through from the one-dimensional case. To be precise, the graph of a solution would be a curve \((r, x(t), y(t))\) in the three-dimensional space, depicting the time evolution of the values of \(x\) and \(y\). We shall use the fact that equations \((5a,b)\) are autonomous to suppress time dependence as before, that is, to depict solutions by trajectories in the plane. Such trajectories, each representing a solution, together make up a phase-plane portrait of the system of equations under consideration.

We observed in Section 5.2 that \((\Delta x, \Delta y)\) given by equations \((7a,b)\) is a vector that depicts both the magnitude and the direction of changes in the two variables. A limiting value of this vector,

\[
\left( \frac{dx}{dt}, \frac{dy}{dt} \right),
\]

(8a)
is obtained when the time increment \(\Delta t\) gets vanishingly small in \((\Delta x/\Delta t, \Delta y/\Delta t)\). The latter, often symbolized

\[
\frac{dx}{dt}
\]

(8b)
represents the instantaneous change in \(x\) and \(y\), and can also be depicted as an arrow attached to the point \((x(t), y(t))\) and tangent to the curve. This vector is often called the velocity vector, since its magnitude indicates how quickly changes are occurring.

A summary of all these facts is collected here:

---

**A Summary of Facts about Vector Functions (from Calculus)**

1. The pair \((x(t), y(t))\) represents a curve in the \(xy\) plane with \(t\) as a parameter.
2. \(x(t) = (x(t), y(t))\) also represents a position vector: a vector attached to \((0, 0)\) that points to the position along the curve, that is, the location corresponding to the value \(t\).
3. The vector \(dx/dt\), which is just the pair \((dx/dt, dy/dt)\) has a well-defined geometric meaning. It is a vector that is tangent to the curve at \(x(t)\). Its magnitude, written \(|dx/dt|\) represents the speed of motion of the point \((x(t), y(t))\) along the curve.
4. The set of equations \((5a,b)\) can be written in vector form,

\[
\frac{dx}{dt} = F(x).
\]

Here the vector function \(F = (f_1, f_2)\) assigns a vector to every location \(x\) in the plane; \(x\) is the position vector \((x, y)\), and \(dx/dt\) is the velocity vector \((dx/dt, dy/dt)\).
Figure 5.6 (a) Point and (b) vector representations of a pair \((x, y)\). (c) A curve \((x(t), y(t))\) can also be represented by moving vector \(x(t)\), as in (d).
5.4 THE DIRECTION FIELD

From concepts that arise in calculus we surmise that solutions to ODEs, whether in one dimension or higher, correspond to curves, and differential equations are "recipes" for tangent vectors to these curves. This insight will now be applied to reconstructing a qualitative picture of solutions to a system of two equations such as (5). For such autonomous systems each point \((x, y)\) in the plane is assigned a unique vector \((f_1(x, y), f_2(x, y))\) that does not change with time. A solution curve passing through \((x, y)\) must have these vectors as its tangents. Thus a collection of such vectors defines a direction field, which can be used as a visual guide in sketching a family of solution curves, collectively a phase-plane portrait. Example 4 clarifies how this is done in practice.

\[
\begin{align*}
\frac{dx}{dt} &= xy - y, \\
\frac{dy}{dt} &= xy - x,
\end{align*}
\]

(9a)
(9b)

and let \(f_1(x, y) = xy - y, f_2(x, y) = xy - x\). In the following table the values of \(f_1\) and \(f_2\) are listed for several values of \((x, y)\).

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>(f_1(x, y))</th>
<th>(f_2(x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>-2</td>
<td>-1</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

After tabulating arbitrarily many values of \((x, y)\) and the corresponding values of \(f_1(x, y)\) and \(f_2(x, y)\), we are ready to construct the direction field. To each point \((x, y)\) we attach a small line segment in the direction of the vector \((f_1(x, y), f_2(x, y))\).
See Figure 5.7. The slope $\Delta y/\Delta x$ of the line segment is to have the ratio $f_2(x, y)/f_1(x, y)$. Notice that a vector $(f_1(x, y), f_2(x, y))$ has the magnitude $\sqrt{f_1(x, y)^2 + f_2(x, y)^2}$, which we shall not attempt to portray accurately. This magnitude represents a rate of motion, the speed with which a trajectory is traced. A cluttered picture emerges should we attempt to draw the vectors $(f_1, f_2)$ in their true sizes. Since we are interested in establishing only the direction field, making all tangent vectors some uniform small size proves most convenient.

![Figure 5.7 Several points $(x, y)$ and the direction vectors $(f_1, f_2)$ associated with them have been sketched above for equations $(9a,b)$.](image)

Two notable locations in example 4 are the points $(0, 0)$ and $(1, 1)$, at both of which $f_1 = 0$ and $f_2 = 0$. Neither $x$ nor $y$ changes given these initial values; the terms steady state, equilibrium point, or singular point are synonymously used to denote such locations. Presently we will see that such points play a central role in determining global phase-plane behavior.

The chore of tabulating and sketching direction fields is in principle straightforward but tedious. Rather than belabor the process we might consign the job to a computer, as we have done in Figures 5.8 $(a,b)$. A simple BASIC program run on an IBM personal computer produced these results.
s to have the ratio has the magnitude accurately. This mag-
jectory is traced. A s \((f_1, f_2)\) in their true
field, making all tan-

\[ \mathbf{\nabla} \]

\[ (g a, b) \]

and \((1, 1)\), at both of
initial values; the
omously used to
ay a central role in
inciple straightfor-
consign the job to a
C program run on

Figure 5.8 (a) Computer-generated vector field for example 4. The vectors point away from the points to which they are attached. For example, along the positive x axis, they point down. (b) Hand-sketched solution curves for example 4. The directions are ascertained by noting whether vectors point into or out of the region at the boundary of the square.
(Computer plot by Yehoshua Keshet.)
From the direction field thus generated one gets a good general idea of solution curves consistent with the flow. Through every point in the plane there is a curve (by existence of a solution) and only one curve (by uniqueness). Thus curves may not intersect or touch each other, except at the steady states designated by heavy dots in Figures 5.7 and 5.8. Rules governing the possible pattern of curves will be outlined in a subsequent section.

As a word of caution, note that a phase-plane diagram is not a quantitatively accurate graph. In practice, because only a finite number of tangent vectors can be drawn in the plane, there will always be some small error in the curve that we inscribe. Such initially small mistakes could propagate if they result in an improper choice of tangent vectors along the way. For this reason, solution curves drawn in this way are approximate. There may be cases where ambiguity arises close to a steady state and where it is difficult to distinguish between several alternatives. Such situations call for a more rigorous technique. Before turning to these matters, we investigate a more systematic way of establishing the direction field in a computationally efficient way.

5.5 NULLCLINES: A MORE SYSTEMATIC APPROACH

Rather than arbitrarily plotting tabulated values, we prepare the way by noticing what happens along the locus of points for which one of the two functions, either \( f_1(x, y) \) or \( f_2(x, y) \) is zero. We observe that

1. If \( f_1(x, y) = 0 \), then \( dx/dt = 0 \), so \( x \) does not change. This means that the direction vector must be parallel to the \( y \) axis, since its \( \Delta x \) component is zero.
2. Similarly, if \( f_2(x, y) = 0 \), then \( dy/dt = 0 \), so \( y \) does not change. Thus the direction vector is parallel to the \( x \) axis, since its \( \Delta y \) component is zero.

The locus of points satisfying one of these two conditions is called a nullcline. The \( x \) nullcline is the set of points satisfying condition 1; similarly, the \( y \) nullcline is the set of points satisfying condition 2. Because the arrows are parallel to the \( y \) and \( x \) axis respectively on these loci, it proves helpful to sketch these as a first step. Example 5 illustrates the procedure.

---

**Example 5**
For equations (9a,b) the nullclines are loci for which

1. \( \dot{x} = 0 \) (the \( x \) nullcline); that is, \( xy - y = 0 \). This is satisfied when \( x = 1 \) or \( y = 0 \). See dotted lines in Figure 5.9(a). On these lines, direction vectors are vertical.
2. \( \dot{y} = 0 \) (the \( y \) nullcline); that is, \( xy - x = 0 \). This is satisfied when \( x = 0 \) or \( y = 1 \). See the dotted-dashed line in Figure 5.9(a). On these lines direction vectors are horizontal.
The general idea of solution one there is a curve (by 
Thus curves may not in 
minated by heavy dots in curves will be outlined 

is not a quantitatively tangent vectors can be 
the curve that we ins 
y result in an improper 
olution curves drawn in 
iguity arises close to a 
veral alternatives. Such 
to these matters, we ins 
field in a computation-

the way by noticing 
etwo functions, either 

This means that the 
$\Delta x$ component is zero. 
not change. Thus the 
ponent is zero.

ns is called a nullcline. 
larly, the $y$ nullcline is 
paralel to the $y$ and $x 
ise as a first step. Exam-

satisfied when $x = 1$ or 
, direction vectors are 
satisfied when $x = 0$ or 
ese lines direction vec-

Figure 5.9 Nullclines and flow directions for 
example 5. (a) Nullclines, which happen to 
be straight lines here, are sketched in the 
xy-plane and assigned vertical or horizontal 
line segments in (b). (c) Directions are 
determined by tabulating several values and 
inscribing arrowheads. (d) Neighboring 
arrows are deduced by preserving a 
continuous flow.

Points of intersection of nullclines satisfy both $\dot{x} = 0$ and $\dot{y} = 0$ and thus repre 
ent steady states. To identify these and determine the directions of flow, several 
guidelines are useful.
Rules for determining steady states and direction vectors on nullclines

1. Steady states are located at intersections of an $x$ nullcline with a $y$ nullcline.
2. At steady states there is no change in either $x$ or $y$ values; that is, the vectors have zero length.
3. Direction vectors must vary continuously from one point to the next on the nullclines. Thus a change in the orientation (for example, from pointing up to pointing down) can take place only at steady states.

We note that $(0, 0)$ and $(1, 1)$ are the only two steady states in example 5. It is important to avoid confusing these with other intersections, for example $(1, 0)$ and $(0, 1)$, for which only one of the two nullcline conditions is satisfied. Generally it is a good idea to distinguish between the $x$ and $y$ nullclines by using different symbols or colors for each type.

It should be remarked that in affixing orientations to the arrows along nullclines we can economize on algebra by being aware of certain geometric properties. For instance, in example 5 we observe the following patterns of signs:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$f_1(x, y)$</th>
<th>$f_2(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-$</td>
<td>1</td>
<td>$-$</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>$-$</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$-$</td>
<td>0</td>
<td>$-$</td>
</tr>
<tr>
<td>0</td>
<td>+</td>
<td>0</td>
<td>$-$</td>
</tr>
<tr>
<td>$+, &gt; 1$</td>
<td>1</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$+, &gt; 1$</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>0</td>
<td>+</td>
<td>$-$</td>
<td>0</td>
</tr>
<tr>
<td>$-$</td>
<td>0</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

It is evident that on opposite sides of a steady-state point (along a given nullcline) the orientation of arrows is reversed. This is a property shared by most systems of equations with the exception of certain singular cases. (We shall be able to distinguish these exceptions by calculating the Jacobian $J$ and evaluating it at the steady state in question. If $\det J \neq 0$, the property of arrow reversal holds.) In most cases where we encounter $\det J \neq 0$, it suffices to determine the direction vectors at one or two select places and deduce the rest by preserving continuity and switching orientation as a steady state is crossed. Thus the arrow-nullcline method can reveal a fairly complete picture with relatively little calculation (see example 6).

**Example 6**
Consider the equations

$$\frac{dx}{dt} = x + y^2,$$

(10a)
with a $y$ nullcline. That is, the vectors to the next on the from pointing up to

example 5. It is implied example (1, 0) and satisfied. Generally it is a different symbols arrows along null-

geometric properties. signs:

(10a)

(along a given null-

shaped by most sys-

We shall be able to evaluating it at the real holds.) In most direction vectors at nuity and switching method can reveal a nple 6).

5.6 CLOSE TO THE STEADY STATES

The examples we have seen give evidence to the notion that dramatic local changes in the flow pattern can only take place in the vicinity of steady-state points. We now invoke a metaphorical magnifying glass to scrutinize the behavior close to these locations. In the discussions of Chapter 4, we established that close to a steady state
Continuous Processes and Ordinary Differential Equations

$(\bar{x}_0, \bar{y}_0)$ [defined by $f_1(\bar{x}_0, \bar{y}_0) = f_2(\bar{x}_0, \bar{y}_0) = 0$] the nonlinear system (5) behaves very nearly like a linear one,

\[
\begin{align*}
\frac{dx}{dt} &= a_{11}x + a_{12}y, \\
\frac{dy}{dt} &= a_{21}x + a_{22}y,
\end{align*}
\]

(11a) (11b)

where $a_{ij}$, related to partial derivatives of $f_1$ and $f_2$, make up the coefficient of the Jacobian matrix $J(\bar{x}_0, \bar{y}_0)$ as follows:

\[
J(\bar{x}_0, \bar{y}_0) = \begin{pmatrix} a_{11} & a_{12} \\
-1 & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \bigg|_{(\bar{x}_0, \bar{y}_0)}
\]

(12)

This result is important, as it reduces the problem to one we understand well. It remains to interpret the phase-plane equivalents of solutions to systems of linear ODEs (described in Chapter 4). This will give us the local picture of the flow pattern about the steady states.

**Example 7**

Equations (9a,b) can be linearized about the steady states $(0, 0)$ and $(1, 1)$. The Jacobian is

\[
J(\bar{x}_0, \bar{y}_0) = \begin{pmatrix} y & x - 1 \\
y & x \end{pmatrix}
\]

One obtains

\[
J(0, 0) = \begin{pmatrix} 0 & -1 \\
-1 & 0 \end{pmatrix}, \quad J(1, 1) = \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}
\]

Thus close to $(0, 0)$ the system behaves much like the linearized version,

\[
\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = -x.
\]

Similarly, close to $(1, 1)$ the linearized equations are

\[
\frac{dx}{dt} = x, \quad \frac{dy}{dt} = y.
\]

A summary of properties of linear systems (of two ordinary differential equations) is given in Table 5.1, in which we consider only the real, distinct eigenvalues case.
### Table 5.1 Linear Systems of two ODEs

<table>
<thead>
<tr>
<th></th>
<th>Full algebraic notation</th>
<th>Equivalent Vector–Matrix Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equations</td>
<td>( \frac{dx}{dt} = a_{11}x + a_{12}y )</td>
<td>( \frac{dx}{dt} = Ax, \quad A = \begin{pmatrix} a_{11} &amp; a_{12} \ a_{21} &amp; a_{22} \end{pmatrix} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{dy}{dt} = a_{21}x + a_{22}y )</td>
<td></td>
</tr>
<tr>
<td>Significant quantities</td>
<td>( \beta = a_{11} + a_{22} ), ( \gamma = a_{11}a_{22} - a_{12}a_{21} ), ( \delta = \beta^2 - 4\gamma )</td>
<td>( \text{Tr} A ), ( \text{det} A ), ( \text{disc} A )</td>
</tr>
<tr>
<td>Characteristic equation</td>
<td>( \lambda^2 - \beta \lambda + \gamma = 0 )</td>
<td>( \text{det} (A - \lambda I) = 0 )</td>
</tr>
<tr>
<td>Eigenvalues</td>
<td>( \lambda_{1,2} = \frac{\beta \pm \sqrt{\delta}}{2} )</td>
<td>( \lambda_{1,2} = \frac{\text{Tr} A \pm \sqrt{\text{disc} A}}{2} )</td>
</tr>
<tr>
<td>Identities</td>
<td>( \lambda_1 + \lambda_2 = \beta )</td>
<td>( \lambda_1 + \lambda_2 = \text{Tr} A, \quad \lambda_1 \lambda_2 = \text{det} A )</td>
</tr>
<tr>
<td>Eigenvectors</td>
<td>( \begin{pmatrix} a_{12} \ \lambda_1 - a_{11} \end{pmatrix}, \begin{pmatrix} a_{12} \ \lambda_2 - a_{11} \end{pmatrix} )</td>
<td>( v_1, v_2 ) such that ( (A - \lambda I)v_i = 0 )</td>
</tr>
<tr>
<td>Solutions</td>
<td>( x = c_1a_{12}e^{\lambda_1t} + c_2a_{12}e^{\lambda_2t} ), ( y = d_1e^{\lambda_1t} + d_2e^{\lambda_2t} ), where ( d_1 = c_1(\lambda_1 - a_{11}), d_2 = c_2(\lambda_2 - a_{11}) )</td>
<td>( x = c_1v_1e^{\lambda_1t} + c_2v_2e^{\lambda_2t} ).</td>
</tr>
</tbody>
</table>
5.7 PHASE-PLANE DIAGRAMS OF LINEAR SYSTEMS

We observe that a linear system can have at most one steady state, at \((0, 0)\) provided \(\gamma = \det A \neq 0\). In the particular case of real eigenvalues there is a rather distinct geometric meaning for eigenvectors and eigenvalues:

1. For real \(\lambda_i\) the eigenvectors \(v_i\) are directions on which solutions travel along straight lines towards or away from \((0, 0)\).
2. If \(\lambda_i\) is positive, the direction of flow along \(v_i\) is away from \((0, 0)\), whereas if \(\lambda_i\) is negative, the flow along \(v_i\) is towards \((0, 0)\).

Proof of these two statements is given below.

An Interpretation of Eigenvectors

Solutions to a linear system are of the form

\[
x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}.
\]

(13)

Recall that \(c_1\) and \(c_2\) are arbitrary constants. If initial conditions are such that \(c_1 = 0\) and \(c_2 = 1\), the corresponding solution is

\[
x(t) = v_2 e^{\lambda_2 t}.
\]

(14)

For any value of \(t\), \(x(t)\) is a scalar multiple of \(v_2\). (This means that \(x(t)\) is always parallel to the direction specified by the vector \(v_2\).) If \(\lambda\) is negative, then for very large values of \(t\), \(x(t)\) is small. In the limit as \(t\) approaches \(+\infty\), \(x(t)\) approaches the steady state \((0, 0)\). Thus \(x(t)\) describes a straight-line trajectory moving parallel to the direction \(v_2\) and towards the origin.

A similar result is obtained when \(c_1 = 1\) and \(c_2 = 0\). Then we arrive at

\[
x(t) = v_1 e^{\lambda_1 t}.
\]

(15)

The solution is a straight-line trajectory parallel to \(v_1\).

It follows that any solution curve that starts on a straight line through \((0, 0)\) in either direction \(\pm v_1\) or \(\pm v_2\) will stay on that line for all \(t\), \(-\infty < t < \infty\) either approaching or receding from the origin. Note also from the above that a steady state can only be attained as a limit, when \(t\) gets infinitely large, because time dependence of solutions is exponential. This tells us that the rate of motion gets progressively slower as one approaches a steady state.

Solution curves that begin along directions different from those of eigenvectors tend to be curved (because when both \(c_1\) and \(c_2\) are nonzero, the solution is a linear superposition of the two fundamental parts, \(v_1 e^{\lambda_1 t}\) and \(v_2 e^{\lambda_2 t}\), whose relative contributions change with time). There is a tendency for the "fast" eigenvectors (those associated with largest eigenvalues) to have the strongest influence on the solutions. Thus trajectories curve towards these directions, as shown in Figure 5.11.
at $(0, 0)$ provided $s$ are rather distinct

ions travel along
$(0, 0)$, whereas if

\begin{equation}
(13)
\end{equation}
such that $c_1 = 0$

\begin{equation}
(14)
\end{equation}
is always paral-
for very large val-
s the steady state
the direction $v_2$
arrive at
\begin{equation}
(15)
\end{equation}
through $(0, 0)$ in
$< t < \infty$ either ap-
that a steady state
time dependence
g gets progressively

of eigenvectors
solution is a linear
ose relative contri-
vector (those as-
the solutions.
ure 5.11.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5_11.png}
\caption{Sketches of the eigenvectors (a–c) and solution curves (d–f) of the linear equations (11a,b) for real eigenvalues. The signs of the two eigenvalues are as follows: (a, d), both positive; (b, e), opposite; (c, f), both negative.}
\end{figure}

Real Eigenvalues

Assuming that eigenvalues are real and distinct (with $\gamma \neq 0$, $\beta^2 - 4\gamma > 0$ where $\beta$, $\gamma$ are as defined in Table 5.1 and equation (16), the behavior of solutions can be classified into one of the three possible categories:

1. Both eigenvalues are positive: $\lambda_1 > 0$, $\lambda_2 > 0$.
2. Eigenvalues are of opposite signs: e.g., $\lambda_1 > 0$, $\lambda_2 < 0$.
3. Both eigenvalues are negative: $\lambda_1 < 0$, $\lambda_2 < 0$.

In these three cases the eigenvectors also are real. Both vectors point away from the origin in case 1 and towards it in case 3. In case 2 they are of opposite orientations, with the one pointing outwards associated with the positive eigenvalue. Figure 5.11(a–c) illustrates this point.
All solutions grow with time in case 1 and decay with time in case 3; hence in each case the point \((0, 0)\) is an **unstable** or a **stable node**, respectively. Case 2 is somewhat different in that solutions approach \((0, 0)\) along one direction and recede from it along another. This unstable behavior is descriptively termed a **saddle point** (see Figure 5.11(e)).

**Complex Eigenvalues**

For \(\lambda = a \pm bi\), we distinguish between the following cases:

4. Eigenvalues have a positive real part \((a > 0)\).
5. Eigenvalues are pure imaginary \((a = 0)\).
6. Eigenvalues have a negative real part \((a < 0)\).

Note that when the linear equations have real coefficients, complex eigenvalues can occur only in conjugate pairs since they are roots of the quadratic characteristic equation.

The eigenvectors are then also complex and have no direct geometric significance. In building up real-valued solutions, recall that the expressions we obtained in Section 4.8 were products of exponential and sinusoidal terms. We remarked on the property that these solutions are oscillatory, with amplitudes that depend on the real part \(a\) of the eigenvalues \(\lambda = a \pm bi\). In the \(xy\) plane, oscillations are depicted by trajectories that wind around the origin. When \(a\) is positive, the amplitude of oscillation grows, so the pair \((x, y)\) spirals away from \((0, 0)\); whereas if \(a\) is negative, it spirals towards it. The case where \(a = 0\) is somewhat special. Here \(e^{at} = 1\), and the amplitude of such solutions does not change. These trajectories are disjoint closed curves encircling the origin, which is then termed a **neutral center**. In this case a somewhat precarious balance exists between the forces that lead to increasing and decreasing oscillations. It is recognized that small changes in a system that oscillates in this way may disrupt the balance, and hence a neutral center is said to be **structurally unstable**. Cases 4, 5, and 6 are illustrated in Figure 5.12.

### 5.8 CLASSIFYING STABILITY CHARACTERISTICS

From certain combinations of the coefficients appearing in the linear equations, we can deduce criteria for each of the six classifications described in the previous section. We shall catalog the nature of the eigenvalues and thus the stability properties of a steady state using three quantities,

\[
\beta = a_{11} + a_{22} = \text{Tr } A, \quad (16a)
\]

\[
\gamma = a_{11}a_{22} - a_{12}a_{21} = \text{det } A, \quad (16b)
\]

\[
\delta = \beta^2 - 4\gamma = \text{disc } A, \quad (16c)
\]

where \(A\) is the \(2 \times 2\) matrix of coefficients \((a_{ij})\) and \(A = J(x_0, y_0)\). [See equation (12)] and \(\text{Tr } (A) = \text{trace}, \text{det } (A) = \text{determinant}, \text{and disc } (A) = \text{discriminant of } A.\)
in case 3; hence inactively. Case 2 is direction and receded a saddle point.

complex eigenvalue
quadratic character-
direct geometric expressions we ob-
dal terms. We re-
amplitudes that de-
plane, oscillations is positive, the am-
(0, 0); whereas if a
what special. Here
ese trajectories are
neutral center. In
ces that lead to in-
hanges in a system
eutral center is said figure 5.12.

inear equations, we
in the previous sec-
ability properties

(16a)
(16b)
(16c)
, \( y_0 \). [See equation
: discriminant of \( A \).
Criteria stem from the fact that eigenvalues are related to these by

$$\lambda_{1,2} = \frac{\beta \pm \sqrt{\delta}}{2}. \quad (17)$$

Consult Figure 5.13 for a graphical interpretation of the arguments that follow.

For real eigenvalues, $\delta$ must be a positive number. Now if $\gamma$ is positive, $\delta = \beta^2 - 4\gamma$ will be smaller than $\beta^2$ so that $\sqrt{\delta} < \beta$. In that case, $\beta + \sqrt{\delta}$ and $\beta - \sqrt{\delta}$ will have the same sign [see Figures 5.13(a) and 5.13(c)]. In other words, the eigenvalues will then be positive if $\beta > 0$ [case 1, Figure 5.13(a)] and negative if $\beta < 0$ [case 3, Figure 5.13(c)]. On the other hand, if $\gamma$ is negative, we arrive at the conclusion that $\sqrt{\delta}$ is bigger than $\beta$. Thus $\beta + \sqrt{\delta}$ and $\beta - \sqrt{\delta}$ will have opposite signs whether $\beta$ is positive or negative [case 2, Figure 5.13(b)].

**Example 8**

In Section 5.6 we saw that the Jacobian of equations (9a,b) for the two steady states $(0, 0)$ and $(1, 1)$ are

$$J(0, 0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad J(1, 1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

Thus for $(0, 0)$, $\beta = 0$ and $\gamma = -1$; so $(0, 0)$ is a saddle point. For $(1, 1)$, $\beta = 2$ and $\gamma = 1$; so $(1, 1)$ is an unstable node.

**Example 9**

Consider the system of equations

$$\frac{dx}{dt} = 2x - y, \quad \frac{dy}{dt} = 3x + 2y.$$  

Then

$$\beta = (2 + 2) = 4, \quad \gamma = (2)(2) + (1)(3) = 7,$$

$$\delta = \beta^2 - 4\gamma = 16 - 28 = -12.$$  

Since $\beta^2 < 4\gamma$, the eigenvalues will be complex. Since $\beta = 4 > 0$, the behavior is that of an unstable spiral.

**Example 10**

Consider the system

$$\frac{dx}{dt} = -4x + y, \quad \frac{dy}{dt} = x - 2y.$$  

Then

$$\beta = (-4 - 2) = -6, \quad \gamma = (-4)(-2) - (1)(1) = 7,$$

$$\delta = \beta^2 - 4\gamma = 36 - 28 = 12.$$  

Since $\beta < 0$ and $\gamma > 0$, the system is a stable node.
Figure 5.13 Eigenvalues are those values $\lambda$ at which the parabola $y = \lambda^2 - \beta \lambda + \gamma$ crosses the $\lambda$ axis. Signs of these values depend on $\beta$ and on the ratio of $\sqrt{\delta}$ to $\beta$ where $\delta = \beta^2 - 4\gamma$. When $\gamma > 0$, both eigenvalues have the same sign as $\beta$. If $\delta < 0$, the parabola does not intersect the $\lambda$ axis, so both eigenvalues are complex.
For eigenvalues to be complex (and not real) it is necessary and sufficient that
\[ \delta = \beta^2 - 4\gamma \] be negative. Then
\[ \lambda = \frac{\beta \pm i\sqrt{-\delta}}{2}. \]

Cases 4, 5, and 6 then follow for positive, zero, or negative \( \beta \) respectively.

To summarize, the steady state can be classified into six cases as follows:

1. Unstable node: \( \beta > 0 \) and \( \gamma > 0 \).
2. Saddle point: \( \gamma < 0 \).
3. Stable node: \( \beta < 0 \) and \( \gamma > 0 \).
4. Unstable spiral: \( \beta^2 < 4\gamma \) and \( \beta > 0 \).
5. Neutral center: \( \beta^2 < 4\gamma \) and \( \beta = 0 \).
6. Stable spiral: \( \beta^2 < 4\gamma \) and \( \beta < 0 \).

The \( \beta\gamma \) parameter plane, shown in Figure 5.14, consists of six regions in
which one of the above qualitative behaviors obtains. This figure captures in a com-

\[ \dot{x} = a_{11}x + a_{12}y, \quad \dot{y} = a_{21}x + a_{22}y, \]

we need only compute the quantities

\[ \beta = a_{11} + a_{22}, \quad \gamma = a_{11}a_{22} - a_{12}a_{21}. \]

The above parameter plane can then be consulted to determine whether the steady state \((0, 0)\) is a node, a spiral point, a center, or a saddle point.

\textbf{Figure 5.14} To get a general idea of what happens in a linear system such as
prehensive way the fundamental characteristics of a linear system. Notice that the region associated with a neutral center occupies a small part of parameter space, namely the positive $\gamma$ axis.

The stability and behavior of a linear system, or the properties of a steady state of a nonlinear system can in practice be ascertained by determining $\beta$ and $\gamma$ and noting the region of the parameter plane in which these values occur. See examples 8, 9, and 10.

5.9 GLOBAL BEHAVIOR FROM LOCAL INFORMATION

Systems of nonlinear ODEs may have multiple steady states (see examples 5 and 6). Close to the steady states, behavior is approximated by the linearized equations, a fact that does not depend on the degree of the system; that is, it holds true in general for $n \times n$ systems.

An attribute of $2 \times 2$ systems that is not shared by those of higher dimensions is that local behavior at steady states can be used to reconstruct global behavior. By this we mean that stability properties of steady states and various gross features of the direction field determine a flow in the plane in an unambiguous way. The reason bigger systems of equations cannot be treated in the same way is that curves in higher dimensions are far less constrained by imposing a continuity requirement. A result that holds in the plane but not in higher dimensions is that a simple closed curve (for example, an ellipse or a circle) separates the plane into two disjoint regions, the “inside” and the “outside.” It can be shown in a mathematically rigorous way that this limits the ways in which curves can form a smooth flow pattern in a planar region. Problem 16 gives some intuitive feeling for why this fact plays such a central role in establishing the qualitative behavior of $2 \times 2$ systems.

The terminology commonly used in the theory of ODEs reflects an underlying analogy between abstract mathematical equations and physical flows. We tend to associate the behavior of solutions to a $2 \times 2$ system with the motion of a two-dimensional fluid that emanates or vanishes at steady-state points. This at least imparts the idea of what a smooth phase-plane picture should look like. (We note a slight exception since saddle points have no readily apparent fluid analogy.) By smooth, or continuous flow we understand that a small displacement from a position $(x_1, y_1)$ to one close to it $(x_2, y_2)$ should not cause a drastic change in the direction of the flow.

There are a limited number of ways that trajectories can be combined to create a flow pattern that accommodates the local (steady-state) properties with the global property of continuity. A partial list follows:

1. Solution curves can only intersect at steady-state points.
2. If a solution curve is a closed loop, it must encircle at least one steady state that cannot be a saddle point (see Chapter 8).

Trajectories can have any one of several asymptotic behaviors (limiting behavior for $t \to +\infty$ or $t \to -\infty$). It is customary to refer to the $\alpha$-limit set and $\omega$-limit set, which are simply the sets of points that are approached along a trajectory for