1. **(15 pts)** Find all eigenvalues and corresponding eigenfunctions for the following Sturm-Liouville problem

\[ y'' + y' + (1 + \lambda)y = 0, \quad y(0) = y(1) = 0. \]

**Solution:**

The eigenvalues are \( \lambda_n = -\frac{3}{4} + n^2 \pi^2, \quad n = 1, 2, \ldots \)

The corresponding eigenfunctions are \( y_n = \alpha_n \sin(n\pi x), \) where \( \alpha_n \neq 0 \) and \( n = 1, 2, \ldots \).

2. **(5 pts)** Consider the piecewise differentiable function \( f(x) \) defined by

\[ f(x) = \begin{cases} 
-2 & -3 \leq x < -1 \\
1 + x^2 & -1 < x \leq 2 \\
2 - x & 2 < x \leq 3 
\end{cases} \]

(a) Sketch the graph of \( f(x) \). Use dots (•) to indicate where a curve segment is defined at an endpoint, and circles (○) to indicate where a curve segment is not defined at an endpoint.

(b) Let \( F(x) \) denote the Fourier series of \( f(x) \) on \([-3, 3]\). Use the Fourier convergence theorem to find the values \( F(-3), F(-1), F(2), \) and \( F(3) \).

**Solution:** \( F(-3) = F(3) = -\frac{3}{2}, \) \( F(-1) = 0, \) and \( F(2) = \frac{5}{2}. \)
(c) Sketch the graph of \( F(x) \) using dots and circles as in (a.).

**Solution:** The graph is the same as the graph of the function except at \( x = -3, x = -1, x = 2 \) and \( x = 3 \), where the value of the Fourier series is the average of the left and right limits at those points.

3. **(10 pts)** Explain, with sufficient detail and by quoting the appropriate results and/or theorem(s), why the following integral formula is **true**:

\[
\int_0^{5\pi} \sin\left(\frac{2x}{5}\right)\sin(3x)dx = 0
\]

(Simply evaluating the integral earns you zero points on this problem).

**Solution:** Consider the following Sturm-Liouville problem on the interval \([0, 5\pi]\): \( X'' + \lambda X = 0 \) with \( X(0) = 0 \) and \( X(5\pi) = 0 \). The eigenfunctions of the problem are \( X_n = c_n \sin\left(\frac{n\pi x}{5}\right) \), and they satisfy the orthogonality relation

\[
\int_0^{5\pi} \sin\left(\frac{n\pi x}{5}\right)\sin\left(\frac{m\pi x}{5}\right)dx = 0 \quad \text{for} \quad m \neq n
\]

Since \( 2 \neq 15 \), the integral in question is simply this orthogonality relation with \( n = 2 \) and \( m = 15 \).

4. **(10 pts)** Find the general solution of the ODE

\[
y'' - \frac{7}{x}y' + (36x^4 - \frac{20}{x^2})y = 0
\]

[HINT: Recall that the following general form: \( y'' - \left(\frac{2n-1}{x}\right)y' + (b^2x^2c^2x^{-2} + a^2x^{-2})y = 0 \). **Solution:** \( y = c_1 x^4 J_2(2x^3) + c_2 x^4 Y_2(2x^3) \)]

5. **(10 pts)** Let \( \{\phi_n(x)\}_{n=1}^{\infty} \) be the eigenfunctions of a **regular** Sturm-Liouville problem on the interval \([a, b]\) corresponding to distinct eigenvalues \( \{\lambda_n\} \). Two eigenfunctions, say \( \phi_m \) and \( \phi_n \) corresponding to eigenvalues \( \lambda_m \) and \( \lambda_n \), will thus satisfy the following differential equations

(a) \( (r(x)\phi_m)' + (q(x) + p(x)\lambda_m) \phi_m = 0 \) ,
(b) \( (r(x)\phi_n)' + (q(x) + p(x)\lambda_n) \phi_n = 0 \)

where the functions \( r(x) \), \( p(x) \) and \( q(x) \) are the same in both equations. Multiplying equation (a) by \( \phi_n \), multiplying equation (b) by \( \phi_m \), and then subtracting one finds that the resulting equation can be put in the form

\[
(\lambda_n - \lambda_m)(p(x)\phi_m \phi_n) = [r(x)(\phi'_m \phi_n - \phi'_n \phi_m)]'
\]

**Starting from this equation** prove that

\[
\int_a^b p(x)\phi_m(x)\phi_n(x)dx = 0 \quad \text{for} \quad n \neq m
\]

for this **REGULAR** Sturm-Liouville problem.

**Solution:** See textbook or your class notes.

6. **(10 pts)** Consider the differential equation

\[
xy'' + 2y' + (x^3 + \lambda x^2)y = 0 \quad x > 0
\]

(a) Write this equation in Sturm-Liouville form and identify the functions \( r(x), p(x) \) and \( q(x) \).

**Solution:** The S-L form of the equation is \( (a^2y')' + (x^4 + \lambda x^2)y = 0 \). Hence \( r(x) = x^2, p(x) = x^3 \) and \( q(x) = x^4 \).

(b) Classify the following three boundary value problems as a

i. **Regular** Sturm-Liouville problem on an appropriate interval
ii. **Periodic** Sturm-Liouville problem on an appropriate interval

iii. **Singular** Sturm-Liouville problem on an appropriate interval

iv. **Non of the above**

**BVP #1**

\[ xy'' + 2y' + (x^3 + \lambda x^2)y = 0 \quad y(1) = y(4), y'(1) = y'(4), \quad 1 \leq x \leq 4 \]

**Solution:** Non of the above. The reason the problem is NOT periodic is that \( r(1) \neq r(4) \).

**BVP #2**

\[ xy'' + 2y' + (x^3 + \lambda x^2)y = 0 \quad y(1) = y(4) = 0, \quad 1 \leq x \leq 4 \]

**Solution:** Regular on \([1,4]\).  

**BVP #3**

\[ xy'' + 2y' + (x^3 + \lambda x^2)y = 0, \quad y \text{ is bounded at } x = 0, \text{ and } y'(4) = 0, \quad 0 \leq x \leq 4 \]

**Solution:** Singular on \([0,4]\).

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**5pts extra credit:**

Let \( f(x) \) be differentiable on the interval \([-1,1]\). Suppose that \( f(x) \) is expanded in a Fourier-Legendre series \( \sum_{n=0}^{\infty} c_n P_n(x) \) and that the following information is known about the coefficients \( c_n \):

\[ c_1 = c_3 = 0, \text{ and also all } c_n = 0 \text{ for } n \geq 5. \]

What can you say about the function \( f(x) \)? That is, describe the function \( f(x) \) as **completely as possible**, based on the above information.

The function is even and contains at least an \( x^4 \) term.

**Take-home exam problems: 35%**

1. Problem #8, page 792.

2. Problem # 14, page 813. Use the following values for the parameters in the problem:

\( a^2 = 4, \ L = \pi, \ \alpha = 2 \), and \( f(x) = (\sin(x))^2 \).