1. (15 pts) Find parameterizations for the following:

(a) The surface that is the portion of the cone \( x^2 + y^2 - z^2 = 0 \) with \( 0 \leq z \leq 9 \), using \( x = u \) and \( y = v \) as the parameters.

The top of the cone where \( 0 \leq z \leq 9 \) is described by \( z = \sqrt{x^2 + y^2} \). Hence we can use \( x \) and \( y \) as parameters.

\[
\vec{r}(x, y) = \langle x, y, \sqrt{x^2 + y^2} \rangle, \quad x^2 + y^2 \leq 81
\]

(b) The surface that is the portion of the sphere \( x^2 + y^2 + z^2 = 4 \) in the first octant.

\[
\vec{r}(\theta, \phi) = \langle 2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi \rangle, \quad 0 \leq \theta \leq \pi/2, \quad 0 \leq \phi \leq \pi/2
\]

(c) The curve that is the graph of \( z = 1 - 2x + x^2 \) in the \( y = -2 \) plane from \((1, -2, 0)\) to \((2, -2, 1)\).

\[
\vec{r}(t) = \langle t, -2, 1 - 2t + t^2 \rangle, \quad 1 \leq t \leq 2
\]

2. (15 pts) One of the following vector fields is conservative and the other is not.

\[
\vec{F}(x, y, z) = (2xy - z)\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 + x + 3z^2)\hat{k}
\]

\[
\vec{G}(x, y, z) = (2xy + z)\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 + x + 3z^2)\hat{k}
\]

(a) Determine which of the vector fields \( \vec{F} \) or \( \vec{G} \) is conservative.

\[
\nabla \times \vec{F} = \langle 0, -2, 0 \rangle \quad \text{and} \quad \nabla \times \vec{G} = \langle 0, 0, 0 \rangle,
\]

so \( \vec{G} \) is conservative. 5 pts

(b) Find all potential functions for the conservative vector field you found in part (a).

\[
f = x^2y + xz + y^2z + z^3 + k
\]

2 pts each factor

3. (20 pts) Set up BUT DO NOT EVALUATE the explicit definite integral needed to compute the line integral of the vector field \( \vec{F}(x, y, z) = (y)\hat{i} + (z)\hat{j} + (x)\hat{k} \) along the curve \( \vec{r}(t) = \langle 5t^3, t, t^2 - 5 \rangle \) for \( 0 \leq t \leq 2 \).

\[
\int_{C} <y, z, x> \cdot d\vec{r} = \int_{0}^{2} <t, t^2 - 5, 5t^3> \cdot <15t^2, 1, 2t> dt
\]

= \int_{0}^{2} (15t^3 + t^2 - 5 + 10t^4) dt

1 pt each

4 pts

4 pts

1 pt

4. (40 pts) Let \( S \) be the parameterized surface given by

\[
\vec{r}(u, v) = \langle u, v, u^2 + v^2 \rangle, \quad 0 \leq u \leq 1, \quad -2 \leq v \leq 2
\]

with the downward orientation.
(a) (10 pts) Compute the normal vector field \( \vec{r}_u \times \vec{r}_v \).

\[
\vec{r}_u \times \vec{r}_v = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
1 & 0 & 2u \\
0 & 1 & 2v \\
\end{vmatrix} = <-2u, -2v, 1 >
\]

(b) (15 pts) Set up (BUT DO NOT EVALUATE) the explicit double iterated integral in \( uv \)-coordinates needed to compute the surface integral \( \iint_S (xyz) \, dS \) over the given surface \( S \).

\[
\iint_S (xyz) \, dS = \int_D \int uv(u^2 + v^2) \sqrt{1 + 4u^2 + 4v^2} \, dA
\]

or

\[
\int_{-2}^2 \int_0^1 uv(u^2 + v^2) \sqrt{1 + 4u^2 + 4v^2} \, dv \, du
\]

(c) (15 pts) Set up (BUT DO NOT EVALUATE) the explicit double iterated integral in \( uv \)-coordinates needed to compute the flux of the vector field \( \vec{F} = < e^y, ye^x, x^2y > \) through the given parameterized surface \( S \) with the downward orientation.

\[
\iint_S < e^y, ye^x, x^2y > \cdot d\vec{S} = -\int_D \int < e^y, ye^x, x^2y > \cdot < -2u, -2v, 1 > \, dA
\]

or

\[
-\int_{-2}^2 \int_0^1 (-2ue^v - 2v^2e^u + u^2v) \, dv \, du
\]

5. (5 pts) Use the fundamental theorem for line integrals to evaluate the line integral

\[
\int_C \nabla f \cdot d\vec{r}
\]

where \( f(x, y, z) = x^2y^3 \sqrt{1 + z^2} \) and \( C \) is any parameterized curve \( \vec{r}(t) \) that starts and ends at \((1, 2, 3)\), and that wraps around the z-axis at least one time.
The fundamental theorem for line integrals is:
\[ \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \]
where \( \vec{r}(a) \) and \( \vec{r}(b) \) are the starting and ending points of the curve, respectively. Since the curve \( C \) starts and ends at the same point, we know that \( \vec{r}(a) = \vec{r}(b) \), and hence the line integral is zero. 5 pts

6. (5 pts) A Maple Question: If you ask Maple V to evaluate the double iterated integral
\[ \int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \sqrt{1+4u^2+4v^2} \, du \, dv \]
you will find that Maple cannot evaluate it exactly. What method would you use (as explained in the 4th Maple lesson) to come to the rescue of Maple to find the exact value of this double iterated integral?

The double iterated integral is over the unit circle \( u^2 + v^2 \leq 1 \) in the \( uv \)-plane, which in polar coordinates is described by \( 0 \leq r \leq 1 \) and \( 0 \leq \theta \leq 2\pi \). Also the integrand can be written as \( \sqrt{1+4(u^2+v^2)} \) which transforms to \( \sqrt{1+r^2} \) in polar coordinates. So I would switch the integral to polar coordinates. 5 pts

• 5 pts EXTRA CREDIT: Answer TRUE or FALSE to the following statement and give a reason for your answer.

"There exists a vector field \( \vec{G} \) such that \( \nabla \times \vec{G} = \langle x^2y, -xy^2, z^3 \rangle \)."

We know the identity \( \nabla \cdot (\nabla \times \vec{G}) = 0 \) for any vector field \( \vec{G} \) that has twice differentiable component functions. But \( \nabla \cdot \langle x^2y, -xy^2, z^3 \rangle = 2xy - 2xy + 3z^2 = 3z^2 \neq 0 \). Hence the answer is FALSE, as no such vector field can exist. 5 pts