Taylor Polynomials

Taylor polynomials are just finite Taylor series. They are useful when we want to approximate functions

\[ T_n(x) = n^{\text{th}} \text{ degree Taylor polynomial} \]

\[ = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!} \]

Technique:
1. Find all derivatives of \( f(x) \) up to \( f^n(x) \)
2. Evaluate \( f \) & its derivatives at \( a \)
3. Plug into
\[ f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!} \]

Ex. 1
Find \( T_2(x) \) for \( f(x) = \sec x \) at \( a = 0 \)

\[ f(x) = \sec x \]
\[ f'(x) = \sec x \tan x \]
\[ f''(x) = (\sec x \tan x) \tan x + \sec x (\sec^3 x) \text{ Product rule!} \]

We just need up to the 2\text{nd} derivative since they want \( T_2 \)
Plug in $a$

$f(a) = \sec a = \frac{1}{\cos a} = 1$

$f'(a) = \sec a \tan a = 1 \cdot 0 = 0$

$f''(a) = (\sec a \tan a) \tan a + \sec^3 a = 0 + 1 = 1$

$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

$T_2(x) = 1 + 0(x-a) + 1(x-a)^2$

$= 1 + \frac{x^2}{2!}$

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$\sec x \approx 1 + \frac{x^2}{2}$ when $x$ is near $0$

Easier to use than $\sec x$

**Ex 2**

(a) Find $T_4(x)$ for $f(x) = \frac{1}{\sqrt{x}}$ at $a = 1$

We need to find up to $f^{(4)}(x)$

- $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$
- $f'(x) = -\frac{1}{2}x^{-3/2}
- f''(x) = -\frac{3}{4}x^{-5/2}$
- $f'''(x) = -\frac{15}{8}x^{-7/2}$
- $f^{(4)}(x) = \frac{15(7)}{16}x^{-9/2}$
\[ f(1) = 1 \\
\quad f'(1) = -\frac{1}{2} \\
\quad f''(1) = \frac{3}{4} \\
\quad f'''(1) = -\frac{15}{8} \\
\quad f^{(4)}(1) = \frac{157}{16} \]

\[ T_4(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \frac{f^{(4)}(a)(x-a)^4}{4!} \]

\[ T_4(x) = 1 - \frac{1}{2}(x-1) + \frac{3}{4}(x-1)^2 + \frac{-15}{8}(x-1)^3 + \frac{157}{16}(x-1)^4 \]

b) Approximate \( \frac{1}{\sqrt{2}} \) with \( T_4(x) \)

We know \( \frac{1}{\sqrt{x}} \approx T_4(x) \) so

\[ \frac{1}{\sqrt{2}} \approx T_4(2) = 1 - \frac{1}{2}(2-1) + \frac{3}{4}(2-1)^2 - \frac{15}{8}(2-1)^3 + \frac{157}{16}(2-1)^4 \]

\[ = 1 - \frac{1}{2} + \frac{3}{4} - \frac{15}{8} + \frac{157}{16} \]

We could have gotten a more accurate approximation by using a higher degree Taylor polynomial.
Ex 3] Find $T_3(x)$ of $f(x) = e^x$, $a = -2$

$\rightarrow f(x) = e^x$
$f'(x) = e^x$
$f''(x) = e^x$
$f'''(x) = e^x$

$\rightarrow f(-2) = e^{-2}$
$f'(-2) = e^{-2}$
$f''(-2) = e^{-2}$
$f'''(-2) = e^{-2}$

Want $T_3$, take up to the 3rd derivative

$\rightarrow T_3(x) = f(a) + f'(a) (x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$

$T_3 = e^{-2} + e^{-2}(x+2) + \frac{e^{-2}(x+2)^2}{2!} + \frac{e^{-2}(x+2)^3}{3!}$