Separable Equations

A separable differential equation is one of the form \( \frac{dy}{dx} = f(y)g(x) \)

To solve it we separate it so on one side we just have x's & on the other side we have y's.

\[
\int \frac{dy}{f(y)} = \int g(x) \, dx
\]

Integrate both sides & add a "+ C" on the side with the independent variable (in this case x).

Solve for y if you can.

Although the process is straightforward, it is equally easy to mess these up.
Examples from Differential Equations by Edwards & Penney

Ex 1. Solve explicitly for $y$

\[ \frac{dy}{dx} = (64xy)^{\frac{1}{3}} \]

* First we need to get this as a function of $x$ multiplied by a function of $y$—the $\frac{1}{3}$ is messing us up

\[ \frac{dy}{dx} = 64^{\frac{1}{3}} x^{\frac{1}{3}} y^{\frac{1}{3}} \]
\[ \frac{dy}{y^{\frac{1}{3}}} = 4 x^{\frac{1}{3}} \, dx \]

* Separate

\[ \int y^{-\frac{1}{3}} \, dy = \int 4 x^{\frac{1}{3}} \, dx \]
\[ \frac{3}{2} y^{\frac{2}{3}} = 4 \left( \frac{3}{4} \right) x^{\frac{4}{3}} + C \]
\[
\frac{3}{2} y^{2/3} = 3x^{4/3} + C
\]

* Here is where people start to mess up. To solve for \( y \), we need to get \( y \) by itself.

\[
y^{2/3} = \frac{2}{3} (3x^{4/3}) + \frac{2}{3} C
\]

\[
y^{2/3} = 2x^{4/3} + C_1
\]

\[
y = (2x^{4/3} + C_1)^{3/2}
\]

**Note:** \( y_0 = (2x^{4/3})^{3/2} + C_1^{3/2} \) is very very wrong.

Likewise so is

\[
y = (2x^{4/3})^{3/2} + C_1
\]
Ex2. Solve the IVP

\[ 2y \frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 16}} \quad y(5) = 2 \]

* Separate

\[
\int 2y \, dy = \int \frac{x}{\sqrt{x^2 - 16}} \, dx
\]

* Need to do a u-sub

\[ u = x^2 - 16 \]
\[ du = 2x \, dx \]
\[ \frac{1}{2} du = x \, dx \]

\[
\int 2y \, dy = \int \frac{1}{\sqrt{u}} \, du
\]
\[ y^2 = \sqrt{u} + C \]
\[ y^2 = \sqrt{x^2 - 16} + C \]
\[ y = \pm \sqrt[4]{x^2 - 16} + C \]
Do determine if we want the + or the −, we need to look at the initial condition
\( y(5) = 2 \). Because we have + 2, we'll use the +

\[
y = \sqrt[2]{x^2 - 16} + C
\]

\[
2 = \sqrt[2]{25 - 16} + C
\]

\[
2 = \sqrt[2]{9} + C
\]

\[
2 = \sqrt[2]{3} + C \quad \rightarrow \quad C = 1
\]

\[
y = \sqrt[2]{x^2 - 16} + 1
\]

* Again, note that when we take a square root of both sides, we can't choose what gets a square root, it needs to be the whole right side.
Ex 3 \[ x \frac{dy}{dx} - y = 2x^2y \] \[ y(1) = 1 \]

* We have a little work to do before it is in the right form

\[ x \frac{dy}{dx} = 2x^2y + y \]

\[ \frac{dy}{dx} = 2x^2y + y \times \frac{1}{x} \]

\[ = y \left( \frac{2x^2 + 1}{x} \right) \]

\[ = y \left( 2x + \frac{1}{x} \right) \]

\[ \int \frac{dy}{y} = \int 2x + \frac{1}{x} \, dx \]

\[ \ln |y| = x^2 + \ln |x| + C \]

\[ |y| = e^{x^2 + \ln |x| + C} \]

\[ |y| = e^{x^2 + \ln |x|} \cdot e^C \]

* Or if we like being fancy

\[ |y| = e^{x^2} e^{\ln |x|} \cdot e^C \]
\[ y = e^{x^2} \times k \]

Remember, \( k = \pm e \), you don't need to write what \( k \) equals on your test.

\[ y = k \times e^{x^2} \]

\[ y(1) = 1 \]

\[ 1 = ke \]

\[ k = \frac{1}{e} \]

\[ y = \frac{1}{e} \times e^{x^2} \quad \text{or} \quad y = xe^{x^2-1} \]

\[ \text{Ex 4} \quad \frac{dy}{dx} = 6e^{2x-y} \]

\( y(0) = 0 \)

* We need to get this in the correct form:

\[ \frac{dy}{dx} = 6e^{2x-y} \]
\[
\frac{dy}{dx} = 6e^{2x}e^{-y}
\]
\[
\int \frac{dy}{e^{-y}} = \int 6e^{2x} \, dx
\]
\[
se^{y}dy = \int 6e^{2x} \, dx
\]
\[
e^{y} = \frac{6}{2}e^{2x} + C
\]
\[
e^{y} = 3e^{2x} + C
\]
\[
y = \ln(3e^{2x} + C)
\]
\[
y(0) = 0
\]
\[
0 = \ln(3e^{0} + C)
\]
*We know \( \ln 1 = 0 \), so we need to set \( 3e^{0} + C = 1 \)
\[
3 + C = 1 \quad \rightarrow \quad C = -2
\]
\[
y = \ln(3e^{2x} - 2)
\]