1. (10 points) Solve the IVP \( y''+10y'+41y=0; y(0)=2, y'(0)=-22 \)

2. (12 points) Find the general solution \( y''+8y'+16y=64x \)

3. (12 points) Use the differential equation \( y''-y'-6y=f(x) \) along with the value of \( f(x) \) listed below to answer the following.
   a) Find the complementary solution, \( y_c \)
   b) Find the form of the particular solution, \( y_p \) but do NOT solve for the coefficients.
      i) \( f(x)=\sin(2x)+6xe^{3x} \)
      ii) \( f(x)=e^{-2x}\sin(3x) \)

4. (10 points) Prove that if \( y_1 \) and \( y_2 \) are both solutions to \( ay''+by'+cy=0 \), then \( c_1y_1+c_2y_2 \) is also a solution.

5. (12 points) A mass with a weight of 8 lb stretches a spring 3 inches. The damping constant is 5. The spring is compressed 6 inches from its equilibrium position and released with no velocity. HINT: Gravity is 32 ft/s²
   a) If \( x(t) \) is the position of the mass at time \( t \), formulate the IVP that describes the motion of the mass.
   b) What kind of damping is this? Justify your answer.

6. (13 points) Use \( a_n = \frac{(-2)^{n+1}}{3^n} \) to answer the following.
   a) Determine if the sequence \( \{a_n\} \) converges or diverges. If it converges, find its limit.
   b) Determine if the series \( \sum_{n=0}^{\infty} a_n \) converges or diverges. If it converges, find its limit.

      Fully justify your answer as we've done in class.

7. (14 points) Determine if the following series converge or diverge, find the sums of convergent series
      Justify your answers thoroughly.
      a) \( \sum_{n=1}^{\infty} \frac{1}{n+2} - \frac{1}{n} \) Include the first 3 partial sums with your answer
      b) \( \sum_{n=1}^{\infty} \frac{3}{2n} \)

8. (10 points) Suppose \( \sum_{n=1}^{\infty} a_n \) converges, show that \( \sum_{n=1}^{\infty} \frac{1}{a_n} \) diverges. Fully justify your work.

9. (7 points) A population of rabbits grows at a rate proportional to its size. Suppose initially, we have 6 rabbits and after 30 days we have 24 rabbits. Find an equation for the number of rabbits after \( t \) days.
1. (10 pts) \( r^2 + 10r + 41 = 0 \)

\[ r = \frac{-10 \pm \sqrt{100 - 4 \cdot 41}}{2} = -5 \pm 4i \]

\[ y = e^{-5x} \left[ C_1 \cos 4x + C_2 \sin 4x \right] \]

\[ y(0) = 2 = C_1 \]

\[ y = e^{-5x} \left[ 2 \cos 4x + C_2 \sin 4x \right] \]

\[ y' = -5e^{-5x} \left[ 2 \cos 4x + C_2 \sin 4x \right] + e^{-5x} \left[ -8 \sin 4x + 4C_2 \cos 4x \right] \]

\[ y'(0) = -22 = -10 + 4C_2 \]

\[ -12 = 4C_2 \quad C_2 = -3 \]

\[ y = e^{-5x} \left[ 2 \cos 4x - 3 \sin 4x \right] \]

2. (12 pts) \( r^2 + 8r + 16 = 0 \)

\( (r+4)^2 = 0 \)

\[ y_c = C_1 e^{-4x} + C_2 xe^{-4x} \]

\[ y_p = Ax + B \]

\[ y_p' = A \]

\[ y_p'' = 0 \quad 0 + 8A + 16Ax + 16B = 64x \]

16A = 64 \quad \rightarrow A = 4

8A + 16B = 0

32 + 16B = 0 \quad \rightarrow B = -2 \quad y_p = 4x - 2

\[ y = C_1 e^{-4x} + C_2 xe^{-4x} + 4x - 2 \]
3. (12 pts)
   a) \( r^2 - r - 6 = 0 \)
   \( (r-3)(r+2) = 0 \)
   \( y_c = C_1 e^{3x} + C_2 e^{-2x} \)
   b) i) \( y_p = A \cos 2x + B \sin 2x + (Ax + B) e^{3x} \)
   ii) \( y_p = e^{-2x} (A \cos 3x + B \sin 3x) \)

4. (10 pts)
   \( y = C_1 y_1 + C_2 y_2 \)
   \( a (c_1 y_1'' + c_2 y_2'') + b (c_1 y_1' + c_2 y_2') + c (c_1 y_1 + c_2 y_2) = 0 \)
   \( ac_1 y_1'' + ac_2 y_2'' + bc_1 y_1' + bc_2 y_2' + cc_1 y_1 + cc_2 y_2 = 0 \)
   \( c_1 (0) + c_2 (0) = 0 \)

5. (12 pts)
   a) \( m x'' + b x' + k x = F_{ext} \)
   \( \frac{1}{4} x'' + 5 x' + 32 x = 0 \)
   \( x (0) = -\frac{1}{2} \)
   \( x' (0) = 0 \)

   b) \( b^2 - 4mk = 25 - 4 \frac{8}{32} \frac{32}{3} < 0 \)
   Under-damping
6. (13 pts)
   a) \( a_n = \frac{(-2)^n(-2)}{3^n} = \left(\frac{-2}{3}\right)^n(-2) \rightarrow 0 \) converges

   b) \( \sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{3^n} = -2 + \frac{(-2)^2}{3} + \frac{(-2)^3}{3^2} + \ldots \)

   \[ a = -2, \quad r = \left(\frac{-2}{3}\right) \]

   \[ |\frac{-2}{3}| < 1 \quad \text{converges} \]

   \[ \frac{a}{1-r} = \frac{-2}{1+\frac{2}{3}} = \frac{-6}{5} = \left[ -\frac{6}{5} \right] \]

7. (14 pts)
   a) Telescoping

   \[ S_1 = \frac{1}{3} - 1 \]
   \[ S_2 = \frac{1}{3} - 1 + \frac{1}{4} - \frac{1}{2} \]
   \[ S_3 = \frac{1}{3} - 1 + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} - \frac{1}{3} \]
   \[ S_4 = -1 + \frac{1}{6} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{4} \]

   \[ S_n = -1 - \frac{1}{2} + \frac{1}{n+1} + \frac{1}{n+2} \]

   \[ \lim_{n \to \infty} S_n = \left[-\frac{3}{2}\right] \quad \text{converges} \]

b) Harmonic series diverges
8. (10pts)

\[ \sum a_n \text{ converges so } \lim_{n \to \infty} a_n = 0 \]

\[ \sum \frac{1}{a_n} \text{ diverges by } \lim_{n \to \infty} \frac{1}{a_n} \to \infty \neq 0 \]

9. (7pts)

\[ y = y_0 e^{kt} \]

\[ y = 6e^{kt} \]

\[ y(30) = 24 = 6e^{30k} \]

\[ 4 = e^{30k} \]

\[ \ln(4) = 30k \]

\[ k = \frac{1}{30} \ln(4) \]

\[ y = 6e^{\frac{1}{30} \ln(4)t} \]