Symmetric Brace Algebras

Tom Lada

This is joint work with Martin Markl.

Recall the definition of a (non-symmetric) brace algebra as given by Gerstenhaber and Voronov:

**Definition 1.** A brace algebra is a graded vector space $U$ together with a collection of degree 0 multilinear braces $x, x_1, \ldots, x_n \mapsto x\{x_1, \ldots, x_n\}$ that satisfy the identities

$$x\{\} = x$$

and

$$x\{x_1, \ldots, x_m\}\{y_1, \ldots, y_n\} =$$

$$\sum \epsilon \cdot x\{y_1, \ldots, y_{i_1}, x_1\{y_{i_1+1}, \ldots, y_{j_1}\}, y_{j_1+1}, \ldots, y_{i_m}, x_m\{y_{i_m+1}, \ldots, y_{j_m}\}, y_{j_m+1}, \ldots, y_n\}$$

where the sum is taken over all sequences $0 \leq i_1 \leq j_1 \leq \cdots \leq i_m \leq j_m \leq n$ and where $\epsilon$ is the Koszul sign of the permutation

$$(x_1, \ldots, x_m, y_1, \ldots, y_n) \mapsto (y_1, \ldots, y_{i_1}, x_1, y_{i_1+1}, \ldots, y_{j_1}, y_{j_1+1}, \ldots, y_{i_m}, x_m, y_{i_m+1}, \ldots, y_{j_m}, y_{j_m+1}, \ldots, y_n)$$

of elements of $U$.

We consider a symmetric version of these braces:

**Definition 2.** A symmetric brace algebra is a graded vector space $B$ together with a collection of degree 0 multilinear braces $x\langle x_1, \ldots, x_n \rangle$ that are graded symmetric in $x_1, \ldots, x_n$ and satisfy the identities

$$x\langle \rangle = x$$

and

$$x\langle x_1, \ldots, x_m \rangle\langle y_1, \ldots, y_n \rangle =$$

$$\sum \epsilon \cdot x\langle x_1\langle y_{i_1}, \ldots, y_{i_1^1}\rangle, x_2\langle y_{i_2^1}, \ldots, y_{i_2^m}\rangle, \ldots, x_m\langle y_{i_m^1}, \ldots, y_{i_m^m}\rangle, y_{i_1^{m+1}}, \ldots, y_{i_m^{m+1}} \rangle$$

where the sum is taken over all unshuffle sequences

$$i_1^1 < \cdots < i_1^1, \ldots, i_1^{m+1} < \cdots < i_1^{m+1}$$

of $\{1, \ldots, n\}$ and where $\epsilon$ is the Koszul sign of the permutation

$$(x_1, \ldots, x_m, y_1, \ldots, y_n) \mapsto (x_1, y_{i_1}, \ldots, y_{i_1^1}, x_2, y_{i_2^1}, \ldots, y_{i_2^m}, \ldots, x_m, y_{i_m^1}, \ldots, y_{i_m^m}, y_{i_1^{m+1}}, \ldots, y_{i_m^{m+1}})$$

of elements of $B$. 1
Some examples with signs suppressed:

Non-symmetric brace relation

\[ x\{x_1, x_2\}\{y\} = x\{y, x_1, x_2\} + x\{x_1, y, x_2\} + x\{x_1, x_2, y\} + x\{x_1\{y\}, x_2\} + x\{x_1, x_2\{y\}\} \]

and the corresponding symmetric brace relation

\[ x\langle x_1, x_2\rangle\langle y\rangle = x\langle y, x_1, x_2\rangle + x\langle x_1\langle y\rangle, x_2\rangle + x\langle x_1, x_2\langle y\rangle\rangle \]

Another example

\[ x\{x_1\}\{y_1, y_2\} = x\{x_1, y_1, y_2\} + x\{y_1, x_1, y_2\} + x\{y_1, y_2, x_1\} + x\{x_1\{y_1\}, y_2\} + x\{y_1, x_1\{y_2\}\} + x\{x_1\{y_1, y_2\}\} \]

and the symmetric version

\[ x\langle x_1\rangle\langle y_1, y_2\rangle = x\langle x_1, y_1, y_2\rangle + x\langle x_1\langle y_1\rangle, y_2\rangle + x\langle x_1\langle y_2\rangle, y_1\rangle + x\langle x_1\langle y_1, y_2\rangle\rangle \]
The motivating example for brace algebra structures is the following: let $V$ be a graded vector space and consider the graded vector space $B_*(V)$ where

$$B_s(V) := \bigoplus_{p-k+1=s} \text{Hom}(V^\otimes k, V)_p$$

where $\text{Hom}(V^\otimes k, V)_p$ denotes the space of $k$-multilinear maps of degree $p$. Given $f \in \text{Hom}(V^\otimes N, V)$ and $g_i \in \text{Hom}(V^\otimes q_i, V)$, where $q_i$ is the degree of $g_i$ as a map, define

$$f\{g_1, \ldots, g_n\} \in \text{Hom}(V^\otimes r, V)_{p+q_1+\ldots+q_n}$$

where $r = a_1 + \ldots + a_n + N - n$ by

$$f\{g_1, \ldots, g_n\} = \sum_{k_0+\ldots+k_n=N-n} (-1)^\beta f(1^\otimes k_0 \otimes g_1 \otimes 1^\otimes k_1 \otimes \ldots \otimes 1^\otimes k_{n-1} \otimes g_n \otimes 1^\otimes k_n),$$

where

$$\beta = \sum_{j<i} (a_i - 1) [k_j + a_j] + \sum_i (N-i) q_i + \sum_{j<i} q_i a_j.$$ 

In this example, suppose that we have a collection of maps

$$\mu_k \in \text{Hom}(V^\otimes k, V)_{k-2} \in B_{-1}(V).$$

If we let $\mu = \mu_1 + \mu_2 + \ldots$, then an $A_\infty$ algebra structure on $V$ may be described by the brace relation $\mu\{\mu\} = 0$. For example (ignoring signs)

$$\mu\{\mu\}(x, y, z) = \mu_1(\mu_3)(x, y, z) + \mu_2(\mu_2)(x, y, z) + \mu_3(\mu_1)(x, y, z) =$$

$$\mu_1(\mu_3(x, y, z)) + \mu_2(\mu_2(x, y), z) + \mu_2(x, \mu_2(y, z))$$

$$+ \mu_3(\mu_1(x, y, z)) + \mu_3(x, \mu(y), z) + \mu_3(x, y, \mu(z)) = 0,$$

i.e. $\mu_2$ is a homotopy associative operation.
We now consider examples of symmetric brace algebras by “symmetrizing” the previous examples. Let $\text{Hom}(V^\otimes k, V)_p^{as}$ be the space of $k$-multilinear maps of degree $p$ that are antisymmetric (or alternating). Consider the graded vector space $B_s(V)$ where

$$B_s(V) := \bigoplus_{p-k+1=s} \text{Hom}(V^\otimes k, V)_p^{as}.$$ 

Given $f \in \text{Hom}(V^\otimes k, V)_p^{as}$ and $g_i \in \text{Hom}(V^\otimes a_i, V)_{q_i}^{as}, 1 \leq i \leq n$, define the symmetric brace $f(g_1, \ldots, g_n) \in \text{Hom}(V^\otimes r, V)_{p+q_1+\cdots+q_n}^{as}$, where $r := a_1 + \cdots + a_n + k - n$, by

$$f(g_1, \ldots, g_n)(v_1, \ldots, v_r) := \sum (-1)^i \chi \cdot f(g_1 \otimes \cdots \otimes g_n \otimes 1^\otimes k-n)(v_{i_1}, \ldots, v_{i_r})$$

with the summation taken over all unshuffles

$$i_1 < \cdots < i_{a_1}, i_{a_1+1} < \cdots < i_{a_1+a_2}, \ldots, i_{a_1+\cdots+a_k+1} < \cdots < i_r,$$

of elements of $V$, where $\chi$ is the antisymmetric Koszul sign of the permutation

$$(v_1, \ldots, v_r) \mapsto (v_{i_1}, \ldots, v_{i_r})$$

and

$$\delta = (k-1)q_1 + (k-2+a_1)q_2 + \cdots + (k-n+a_1+\cdots+a_{n-1})q_n$$

$$+ \sum_{1 \leq i < j \leq n} a_i a_j + (n-1) a_1 + (n-2) a_2 + \cdots + a_{n-1}.$$ 

In this example, suppose that we have maps

$$l_k \in \text{Hom}(V^\otimes k, V)_{k-2}^{as} \in B_{-1}(V).$$

If we let $l = l_1 + l_2 + \ldots$, then an $L_\infty$ algebra structure on $V$ is given by the symmetric brace relation $l(l) = 0$.

Example (again, ignoring signs): $l(l)(x, y, z) =$

$$l_1(l_3)(x, y, z) + l_2(l_2)(x, y, z) + l_3(l_1)(x, y, z) =$$

$$l_1(l_3(x, y, z)) + l_2(l_2(x, y), z) + l_2(l_2(x, z), y) + l_2(l_2(y, z), x)$$

$$l_3(l_1(x), y, z) + l_3(l_1(y), x, z) + l_3(l_1(z), x, y) = 0.$$
Symmetrization Theorems

1. Given a brace algebra structure on the graded vector space $U$, then
   
   \[ f(g_1, \ldots, g_n) = \sum_{\sigma \in S_n} \epsilon \cdot f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\} \]

   is a symmetric brace algebra structure on $U$ where $\epsilon$ is the Koszul sign of the permutation $\sigma$.

2. Given $f : V^\otimes n \to V$, and let $as(f)$ denote the map
   
   \[ as(f)(v_1, \ldots, v_n) = \sum_{\sigma \in S_n} \chi f(v_{\sigma(1)}, \ldots, v_{\sigma(n)}). \]

   Let $f, g_1, \ldots, g_n \in Hom(V^\otimes*, V)$. Then
   
   \[ \sum_{\sigma \in S_n} \epsilon \cdot as(f\{g_{\sigma(1)}, \ldots, g_{\sigma(n)}\}) = as(f)(as(g_1), \ldots, as(g_n)). \]

3. Corollary: Consider the symmetric brace algebra structure on $\bigoplus Hom(V^\otimes*, V)$ given by the symmetrization of the non-symmetric brace structure given earlier. Let $\bigoplus Hom(V^\otimes*, V)^{as}$ have the symmetric brace algebra structure given above. Then
   
   \[ as : \bigoplus Hom(V^\otimes*, V) \to \bigoplus Hom(V^\otimes*, V)^{as} \]

   is a homomorphism of symmetric brace algebras.

4. Also as a Corollary, we have
   
   The anti-symmetrization $l := as(\mu)$ of an $A_\infty$ algebra structure $\mu$ yields an $L_\infty$ algebra structure.

   Proof: Given $\mu\{\mu\} = 0$. We have
   
   \[ 0 = as(\mu\{\mu\}) = as(\mu)(as(\mu)) = l(\langle l \rangle). \]
MOTIVATION FROM A PHYSICS PROBLEM

So let $L$ be the graded vector space with $L_0 = \Xi$, $L_{-1} = \Phi$, $L_n = 0$, $n \neq 0, -1$.

Consider the symmetric brace algebra structure on $B_*(L) = \text{Hom}(L^\otimes^*, L)^{\text{as}}$.

Construct a bracket on $\text{Hom}(L^\otimes^*, L)^{\text{as}}$ so that the existence of an $L_0$ structure on $L$ is equivalent to this bracket satisfying the Jacobi identity on $\text{Hom}(\Phi^\otimes^*, \Xi)^{\text{as}}$.

Fix two maps, $\nabla$ and $\Upsilon$ in $\text{Hom}(L^\otimes^*, L)^{\text{as}}$ we require that

1) $\nabla$ has values in $\Phi$ and is the zero map when the number of inputs from $\Xi$ is not equal to 1.

2) $\Upsilon$ takes values in $\Xi$ and is the zero map when the number of inputs from $\Xi$ is not equal to 2.

E.g. $\text{Hom}(\Xi, \text{Hom}(\Lambda^* \Phi, \Phi)) \ni \delta \mapsto \nabla \in \text{Hom}(\Xi \otimes \Lambda^* \Phi, \Phi)$

$\text{Hom}(\Xi \otimes \Xi, \text{Hom}(\Lambda^* \Phi, \Xi)) \ni \gamma \mapsto \Upsilon \in \text{Hom}(\Xi \otimes \Xi \otimes \Lambda^* \Phi, \Xi)$

Note that $\nabla, \Upsilon \in B_{-1}(L)$. For $\alpha, \beta \in B_*(L)$ define a degree 1 bracket

$[\alpha, \beta] = \nabla \langle \alpha \langle \beta \rangle \rangle + (-1)^{||\alpha||} \beta \langle \nabla \langle \alpha \rangle \rangle + \Upsilon \langle \alpha, \beta \rangle.$
Theorem (L.M)

The above bracket, restricted to \( \text{Hom}(\mathcal{F} \otimes \Xi) \subset \text{B}_c(\nu) \) satisfies the Jacobi identity if and only if

\[
\bigtriangleup \langle a \rangle + \bigtriangledown \langle r \rangle = 0
\]

and \( \gamma \langle a \rangle + \gamma \langle r \rangle = 0 \)

Cor: if this bracket satisfies the Jacobi identity, then the map \( l = \bigtriangleup + \bigtriangledown \) gives an \( \mathcal{L}_o \) structure for \( L \)

Proof:

\[
\langle l \rangle = (\bigtriangleup + \bigtriangledown) \langle \bigtriangleup + \bigtriangledown \rangle
\]

\[
= (\bigtriangleup \langle a \rangle + \bigtriangledown \langle r \rangle) + (\gamma \langle a \rangle + \gamma \langle r \rangle)
\]

\[
= 0
\]