Orbifold Cohomology for Global Torus Quotients

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**WARNING:** These are lecture notes!

**Introductory Example.**

Consider the topological quotient of $S^3$ by $S^1$

where the action is:

$$S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \}$$

$S^1$ acts on $S^3$ by:

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^3 z_2).$$

The quotient space $S^3/S^1$ is topologically a 2-sphere, but:

The orbit of points $(2, 0)$ form the North pole, with isotropy $\mathbb{Z}_2$:

1, -1 both fix $(2, 0)$

The orbit of a generic point $(z_1, z_2)$
$(z_1, z_2 \neq 0)$ is smooth in the quotient.

it really looks like a lemon.

The orbit of $\infty$ pts $(0, z_2)$ form the South pole, with isotropy $\mathbb{Z}_3$:

1, $e^{2\pi i/3}$, $e^{4\pi i/3}$

all fix $(0, z_2)$. 
The lemon is not the global quotient of any space by a finite group.

The inertial cohomology

How can we devise an invariant to tell the difference between the smooth $S^2$ and the lemon?

Introduce the inertial ring.

Let $M$ be a manifold with a locally free $T$ action (here, $T$ is a torus, and loc. free means finite isotropy).

Then as a (graded) vector space:

$$NH^*_{T^0}(M) := \bigoplus_{g \in T} H^*(M^g/T)$$

$$M^g = \{ m \in M \mid g \cdot m = m \}$$

(Many $M^g = \emptyset$)

- The grading $*$ is shifted from left hand side to right hand side. We won't discuss this.

- The grading $\oplus$ is over the group elements. We won't discuss this either (both gradings are related to the product structure, which we won't have time for).
In the case of \( M = S^3 \) and \( T = S' \) with the action as before,

\[
NH^*: (S^3) = \frac{H^*(S^3)}{g=1} \oplus \frac{H^*(i(z,0,0)\frac{1}{3})}{g=-1} \oplus \frac{H^*(i(0,1,2)\frac{1}{3})}{g = e^{2\pi i/3}} \oplus \frac{H^*(i(0,2,1)\frac{1}{3})}{g = e^{4\pi i/3}}
\]

**Equivariant Cohomology**

\[
\mathcal{H}^*_T(M) := H^*\left(\frac{M \times ET}{T}\right)
\]

where ET is a contractible space on which \( T \) acts freely, and \( M \times ET := (M \times ET) / T \)

Note: \( \mathcal{H}^*_T(M) = H^*(M / T) \) when \( T \) acts freely.

We may rewrite:

\[
NH^*_{\mathcal{T}}(M) := \bigoplus_{g \in T} H^*_T(M^g)
\]

(when \( T \) acts on \( M \) locally freely).
Chen & Ruan describe a funny ring structure on this space, but it's not in practice easy to compute.

Idea: move away from locally free actions

Let $Y$ be a Hamiltonian $T$-space. ($T = \text{compact abelian Lie group}$)

This means $Y$ is symplectic ($\omega$ has a closed, non-degenerate 2-form $\omega$ on it).

and if $\xi \in \text{Lie}(T)$ is a vector in the tangent space, it generates the vector field $X_\xi$ on $Y$, then $\exists$ function $\phi^\xi$ on $Y$ such that

$$\omega(X_\xi, \cdot) = d\phi^\xi.$$

In particular $\phi : Y \to \text{Lie}(T)^*$ defined by

$$\langle \phi(p), \xi \rangle \coloneqq \phi^\xi(p)$$

is a moment map for $T \acts Y$.

Hamiltonian $T$-spaces do not have locally free actions: if $Y$ is compact, the functions $\phi^\xi$ have max/min on $Y$.

$\Rightarrow \ \text{df}_\phi^\xi = 0$ at these pts

$\Rightarrow$ by nondegen. $\omega$, $X_\xi = 0$ at these points

$\Rightarrow$ the $ST$ generated by (rational) $\xi$ fixes the crit. point of $\phi^\xi$. 
However, the inertial cohomology ring is still well-defined using equivariant cohomology

\[ NH^*_T(Y) := \bigoplus_{g \in G} H^*_T(Y^g) . \]

In the case that \( T \simeq Y \) locally freely, this is the Chen-Ruan cohomology of \( Y_T \).

In the case \( q : T \simeq Y \) in a Hamiltonian fashion, this is something else.

If \( T \simeq Y \) in a Hamiltonian fashion with fixed points \( Y^T \), then

\[ H^*_T(Y) \rightarrow H^*_T(Y^T) \text{ is an INJECTION.} \]

If \( Y^T \) is isolated,

\[ H^*_T(Y) \rightarrow \bigoplus_{p \in Y^T} H^*_T(p) \]

\[ = \bigoplus_{p \in Y^T} \mathbb{Q}[u_1, \ldots, u_d] \text{ where } d = \dim T \]

\[ (\text{since } H^*_T(p) = H^*_T(p^T \cap T) = H^*_T(\mathbb{C}P^d)) \]

Contrast this to ordinary cohomology where there is only degree 0 cohomology on \( Y^T \), when isolated.

\[ \text{AND NO POSSIBLE INJECTION IN GENERAL.} \]
Let \( \phi : Y \to \text{Lie}(T)^* \)
be a moment map. Assume \( 0 \) is a regular value.
Then \( \phi^{-1}(0) \cap Y \) is a submanifold
with a locally free \( T \)-action.

**Ex**

\( Y = S^2 \), \( \omega \) = volume form

\( S' \cap Y \) by spinning it on \( z \)-axis, fixing \( N \) and \( S \) poles

\(
\begin{align*}
\phi : Y &\to \mathbb{R} \\
S^2 &\xrightarrow{\text{ht function}} \\
&\begin{cases} 1 \\
-1 \end{cases}
\end{align*}
\)

\( 0 \) is a regular value.
\( \phi^{-1}(0) \) is equator. It has a free \( S' \)-action.

The symplectic reduction

\( Y//T := \phi^{-1}(0)/T \)

is an orbifold.
Theorem (G.-H. Holm, Knutson): There is a surjection
\[ NH^*_\mathbb{T}(Y) \rightarrow NH^*_\mathbb{T}(\phi^{-1}(0)) \]
\[ \cong H^*_{CR}(Y//\mathbb{T}) \]
as rings from the inertial cohomology
of a Hamiltonian \( \mathbb{T} \)-space \( Y \) to
the Chen-Ruan cohomology of the symplectic quotient \( Y//\mathbb{T} \).

If \( Y = T^*\mathbb{C}^n \)
is Hyperkähler, and \( Y//\mathbb{T} \)
is a hypertoric variety, then

Thm (G.-H. Harada),
\[ NH^*_\mathbb{T}(Y) \rightarrow H^*_{CR}(Y//\mathbb{T}) \]
is also a surjection.

By the way, the lemon (our original example)
is a symplectic reduction

of \( \mathbb{C}^2 \) by an \( S' \) action:

\[ \lambda: (z_1, z_2) \mapsto (\lambda^2 z_1, \lambda^3 z_2). \]
The moment map \( \phi: \mathbb{C}^2 \rightarrow \mathbb{R} \) is given by

\[ (z_1, z_2) \mapsto |z_1|^2 + |z_2|^2 - 1. \]

Then lemon = \( \phi^{-1}(0)/S' \).