The $A_\infty$ Matrad and the Polytopes $KK$

The Polytope $KK_{2,4}$

Joint work with

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October 29, 2006
Goal: Specify the combinatorics of $KK_{n,m}$ in dimensions $\leq 3$

- $\dim KK_{n,m} = n + m - 3$
- $KK_{n,1} = KK_{1,n} = K_n$ is Stasheff’s associahedron

Strategy: Define the cellular boundary $\partial$ on top dim’l faces and extend inductively to lower dimensions.

- Matrad generators are double corollas

\[ \theta^n_m = \begin{array}{c} n \\ \cdots \\ m \end{array} \]

thought of as operations in a PROP

\[ M = \{ M_{n,m} = \text{Hom}(H^\otimes m, H^\otimes n) \} \]

- Data flows upward
Markl’s Fraction Product

Monomials generated by the $\theta_{m}^{n}$’s have form

$$\alpha_{x}^{y} = \frac{\alpha_{p}^{y_{1}} \cdots \alpha_{p}^{y_{q}}}{\alpha_{x_{1}}^{q} \cdots \alpha_{x_{p}}^{q}},$$

where

1. $x = \sum x_{i}$ and $y = \sum y_{j}$
2. # of outputs from each factor below is # of factors above
3. # of inputs to each factor above is # of factors below
4. The $j^{th}$ output of the $i^{th}$ factor below links to the $i^{th}$ input of $j^{th}$ factor above
The operad of up-rooted Planar Rooted Trees (PRTs) is the free associative algebra

\[ A = FA ( | , \big\uparrow , \big\uparrow , \big\downarrow , \big\downarrow , \ldots ) \]

Monomials in \( A \) are fraction products:

\[ \partial ( \theta_{m}^{1} ) = \sum_{p=m-1}^{\infty} \alpha_{p}^{1} \in A \cdot A \] (we ignore signs)

Similarly, in the operad of down-rooted PRTs

\[ \partial ( \theta_{m}^{n} ) = \sum_{q=n-1}^{\infty} \alpha_{q}^{1} \in C \cdot C, \]

where \( C = FA ( | , \big\uparrow , \big\uparrow , \big\downarrow , \big\downarrow , \ldots ) \)

Define \( \partial ( \theta_{m}^{n} ) \) by appropriately restricting the fraction product to a matrad product.
The Matrad Product in dimensions $\leq 3$

The lower and upper leaf sequences of a monomial

$$\alpha^y_x = \frac{\alpha_{x_1}^{y_1} \cdots \alpha_{x_p}^{y_p}}{\alpha_{x_1}^{q_1} \cdots \alpha_{x_p}^{q_p}}$$

$$LLS(\alpha^y_x) = (x_1, \ldots, x_p)$$

$$ULS(\alpha^y_x) = (y_1, \ldots, y_q)$$

Example: For $\alpha^5_6 = \begin{array}{c|c|c}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array}$

$$LLS(\alpha^5_6) = (2, 1, 3) ; \: ULS\left(\alpha^5_6\right) = (3, 2) .$$

The lower and upper contact sequences of $\alpha^5_6$ are the lists of lower and upper leaf sequences

$$((2), (2), (2)) \: \text{and} \: ((2, 1), (3)) .$$
S-U Diagonal on Associahedra

Define $\Delta_K : C_* \left( K_n \right) \to C_* \left( K_n \right) \otimes C_* \left( K_n \right)$ by

$$\Delta_K \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right) = \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}$$

$$\Delta_K \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right) = \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}$$

$$\Delta_K \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right) = \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}$$

$$\begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \otimes \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array}$$

Define the *left-iterated diagonal* via

$$\Delta_K^{(0)} = \text{id}$$

$$\Delta_K^{(k)} = \left( \Delta_K \otimes \text{id} \otimes \cdots \otimes \text{id} \right) \Delta_K^{(k-1)}$$
- View each component of $\Delta^{(k)}_K$ as an $(p - 2)$-dim’l subcomplex of $K_p \times K^{k+1}$.

- A non-vanishing matrad monomial

$$\alpha^y_x = \frac{\alpha^{y_1}_p \cdots \alpha^{y_q}_p}{\alpha^{q}_{x_1} \cdots \alpha^{q}_{x_p}}$$

in dimension $\leq 3$ satisfies

1. The lower contact sequence of $\alpha^y_x$ agrees with the list of ULS’s of some component of $\Delta^{(p-1)}_K$.

2. The upper contact sequence of $\alpha^y_x$ agrees with the list of LLS of some component of $\Delta^{(q-1)}_K$. 

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Example: In
\[
\alpha_6^5 = \begin{array}{c}
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\end{array}
\end{array},
\]
the lower contact sequence \(((2), (2), (2))\) is the list of ULS’s of
\[
Y \otimes Y \otimes Y = \Delta_{K}^{(2)}(Y)
\]
and the upper contact sequence \(((2, 1), (3))\) is the list of LLS’s of
\[
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\end{array}
\end{array} \text{ in } \Delta_{K}^{(1)}(\begin{array}{c}
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\end{array}
\end{array}).
\]

For \(m + n \leq 6\), define
\[
\partial (\theta_{m}^{n}) = \sum_{p+q=m+n-1} \alpha_{p}^{q} \in B \cdot B,
\]
where \(B = FA(\ \begin{array}{c}
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\]
\[ \partial(X) = \frac{\chi}{\lambda} + \frac{\lambda}{\chi} \]

\[ \partial(\chi) = \frac{\chi}{\lambda} + \frac{\chi}{\lambda} + \frac{\chi}{\lambda} + \frac{\chi}{\lambda} + \frac{\chi}{\lambda} + \frac{\chi}{\lambda} + \frac{\chi}{\lambda} + \frac{\chi}{\lambda} \]
\[ \partial \left( \eta \right) = \eta + \frac{\eta}{\eta} + \frac{\eta}{\eta\eta} + \frac{\eta}{\eta\eta\eta} + \frac{\eta}{\eta\eta\eta\eta} + \frac{\eta}{\eta\eta\eta\eta\eta} + \frac{\eta}{\eta\eta\eta\eta\eta\eta} + \frac{\eta}{\eta\eta\eta\eta\eta\eta\eta} + \frac{\eta}{\eta\eta\eta\eta\eta\eta\eta\eta} + \frac{\eta}{\eta\eta\eta\eta\eta\eta\eta\eta\eta} + \frac{\eta}{\eta\eta\eta\eta\eta\eta\eta\eta\eta\eta} + \frac{\eta}{\eta\eta\eta\eta\eta\eta\eta\eta\eta\eta\eta} + \frac{\eta}{\eta\eta\eta\eta\eta\eta\eta\eta\eta\eta\eta\eta} + \frac{\eta}{\eta\eta\eta\eta\eta\eta\eta\eta\eta\eta\eta\eta\eta} \]

- Note iterated \( \Delta_K \) in “numerator/denominator”
$KK_{3,3}$:
$A_\infty$-bialgebras

- The bialgebra matrad $\mathcal{H}_\infty$ is realized by $C_\ast(KK)$

- $\text{End}(TH)$ is canonically a matrad

- A map of matrads $\mathcal{H}_\infty \rightarrow \text{End}(TH)$ imposes an $A_\infty$-bialgebra structure on an $R$-module $H$.

- An $A_\infty$-bialgebra is an algebra over $\mathcal{H}_\infty$

An alternative...
The Biderivative

Choose arbitrary

\[ \omega^n_m \in M_{n,m} = \text{Hom}(H^\otimes m, H^\otimes n) \]

and consider \( \omega = \sum_{m,n \geq 1} \omega^n_m \)

The “biderivative” \( d_\omega : M \to M \) induces a non-bilinear operation

\[ \odot : M \times M \xrightarrow{d \times d} M \times M \xrightarrow{\pi} M \xrightarrow{\text{proj}} M \]

Alternative Definition:

\( (H, \omega) \) is an \( A_\infty \)-bialgebra if \( \omega \odot \omega = 0 \)
Construct $d_\omega$ as follows:

- Linearly extend $d = \omega_1^1$ to $(H \otimes p) \otimes q$

- Freely extend the map
  \[ \sum_{j \geq 1} \omega_1^j : H \to T^a H \]
as a derivation

- Cofreely extend the map
  \[ \sum_{i \geq 1} \omega_i^1 : T^c H \to H \]
as a coderivation

- Freely extend the map
  \[ \sum_{j > 1} \omega_i^j : H^\otimes i \to T^a H \]
as a $\Delta_P$-derivation for each $i$

- Cofreely extend the map
  \[ \sum_{i > 1} \omega_i^j : T^c H \to H^\otimes j \]
as a $\Delta_P$-coderivation for each $j$
To picture this, make the identification

\[
(H \otimes p) \otimes q \leftrightarrow (p, q) \in \mathbb{N}^2
\]

and represent \(\omega^q_p : H \otimes p \rightarrow H \otimes q\) as a “transgressive” arrow \((p, 1) \rightarrow (1, q)\):

- Represent components \(A\) and \(B\) of the extensions above as arrows in \(\mathbb{N}^2\)
• When the terminal point of $B$ is the initial point of $A$, the (restricted) fraction product $\gamma$ is given by

$$\gamma(A \otimes B) = A \circ \sigma_{p,q} \circ B,$$

where $\sigma_{p,q} : (H^\otimes p)^q \approx (H^\otimes q)^p$ is the canonical permutation of tensor factors.

• Define $\omega \otimes \omega = \sum_{A,B \in d_\omega} \gamma(A \otimes B)$

• Summands of $\omega \otimes \omega$ have one of two types:

  1. $\gamma(\omega^k_j \otimes (1 \cdots \omega^1_i \cdots 1))$ and vise versa

  2. $\gamma(A_1 \cdots A_t \otimes B_1 \cdots B_s)$ with $s, t \geq 2$

  (sequences of arrows)
• When $\omega \otimes \omega = 0$, there is a relation involving certain “transgressive” products $\gamma (A \otimes B)$ from $(p, 1)$ to $(1, q)$ for each $p$ and $q$.

**Example:** When $\omega_2^2 = 0$ there is the relation

$$d \times + \times d = \times + \bigotimes + \bigotimes$$