A FINITE DIMENSIONAL $A_\infty$ ALGEBRA EXAMPLE

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Dedicated to Tornike Kadeishvili on the occasion of his 60th birthday

Abstract. We construct an example of an $A_\infty$ algebra structure defined over a finite dimensional graded vector space.

Introduction

$A_\infty$ algebras (or sha algebras) and $L_\infty$ (or sh Lie algebras) have been topics of current research. Construction of small examples of these algebras can play a role in gaining insight into deeper properties of these structures. These examples may prove useful in developing a deformation theory as well as a representation theory for these algebras.

In [2], an $L_\infty$ algebra structure on the graded vector space $V = V_0 \oplus V_1$ where $V_0$ is a 2 dimensional vector space, and $V_1$ is a 1 dimensional space, is discussed. This surprisingly rich structure on this small graded vector space was shown by Kadeishvili and Lada, [3], to be an example of an open-closed homotopy algebra (OCHA) defined by Kajiura and Stasheff [4]. In an unpublished note [1] M. Daily constructs a variety of other $L_\infty$ algebra structures on this same vector space.

In this article we add to this collection of structures on the vector space $V$ by providing a detailed construction of non-trivial $A_\infty$ algebra data for $V$.

1. $A_\infty$ Algebras

We first recall the definition of an $A_\infty$ algebra (Stasheff [6]).

Definition 1.1. Let $V$ be a graded vector space. An $A_\infty$ structure on $V$ is a collection of linear maps $m_k : V^{\otimes k} \rightarrow V$ of degree $2 - k$ that satisfy the identity

$$
\sum_{\lambda=0}^{n-1} \sum_{k=1}^{n-\lambda} \alpha m_{n-k+1}(x_1 \otimes \cdots \otimes x_{\lambda} \otimes m_k(x_{\lambda+1} \otimes \cdots \otimes x_{\lambda+k}) \otimes x_{\lambda+k+1} \otimes \cdots \otimes x_n) = 0
$$

where $\alpha = (-1)^{k+\lambda+k\lambda+k\lambda+k(|x_1|+\cdots+|x_\lambda|)}$, for all $n \geq 1$.

This utilizes the cochain complex convention. One may alternatively utilize the chain complex convention by requiring each map $m_k$ to have degree $k - 2$.
We will define the desuspension of $V$ (denoted $\downarrow V$) as the graded vector space with indices given by $(\downarrow V)_n = V_{n+1}$, and the desuspension operator, $\downarrow : V \to \downarrow V$ (resp. suspension operator $\uparrow : \uparrow V \to V$) in the natural sense.

Stasheff also showed that an $A_\infty$ structure on $V$ is equivalent to the existence of a degree 1 coderivation $D : T^* \downarrow V \to T^* \downarrow V$ with the property $D^2 = 0$. Here, $T^* \downarrow V$ is the tensor coalgebra on the graded vector space $\downarrow V$.

Such a coderivation is constructed by defining

$$D := \sum_{k=1}^{\infty} m'_k,$$

where $m'_k : \downarrow V^\otimes k \to \downarrow V$ is given by first defining $m'_k := (-1)^{k(k-1)/2} \downarrow m_k \circ \uparrow^\otimes k$ and then extending each $m'_k$ to a coderivation on $T^* \downarrow V$.

2. A Finite Dimensional Example

Let $V$ denote the graded vector space given by $V = \bigoplus V_n$ where $V_0$ has basis $< v_1, v_2 >$, $V_1$ has basis $< w >$, and $V_n = 0$ for $n \neq 0, 1$. Define a structure on $V$ by the following linear maps $m_n : V^\otimes n \to V$:

$$m_1(v_1) = m_1(v_2) = w$$

For $n \geq 2$:

$$m_n(v_1 \otimes w^\otimes k \otimes v_1 \otimes w^\otimes (n-2)-k) = (-1)^k s_n v_1, \quad 0 \leq k \leq n-2$$

$$m_n(v_1 \otimes w^\otimes (n-2) \otimes v_2) = s_{n+1} v_1$$

$$m_n(v_1 \otimes w^\otimes (n-1)) = s_{n+1} w$$

where $s_n = (-1)^{(n+1)(n+2)/2}$, and $m_n = 0$ when evaluated on any element of $V^\otimes n$ that is not listed above.

Theorem 2.1. The maps defined above give the graded vector space $V$ an $A_\infty$ algebra structure.

It is worth noting that this assumes the cochain convention regarding $A_\infty$ algebra structures. The proof of this theorem relies on two lemmas:

Lemma 2.2. Let $m'_n : \downarrow V^\otimes n \to \downarrow V := (-1)^{n(n-1)/2} \downarrow m_n \circ \uparrow^\otimes n$ where $\downarrow V$ denotes the desuspension of $V$. Under the preceding definitions for $m_n$ and $V$, we have the following definitions for $m'_n$:

$$m'_1 = \downarrow m_1$$

For $n \geq 2$:

$$m'_n(\downarrow v_1 \otimes \downarrow w^\otimes k \otimes \downarrow v_1 \otimes \downarrow w^\otimes (n-2)-k) = \downarrow v_1, \quad 0 \leq k \leq n-2$$

$$m'_n(\downarrow v_1 \otimes \downarrow w^\otimes (n-2) \otimes \downarrow v_2) = \downarrow v_1$$

$$m'_n(\downarrow v_1 \otimes \downarrow w^\otimes (n-1)) = \downarrow w$$
Remark 2.3. Each $m'_n$ is of degree 1.

Lemma 2.4. Let $D = \sum_{k=1}^{\infty} m'_k$ where $m'_k$ is defined above. Let $n \geq 2$ be a positive integer. Suppose $D^2(\downarrow x_1 \downarrow x_2 \cdots \downarrow x_m) = 0 \forall x_i \in V, 1 \leq m \leq n - 1$.

Then $D^2(\downarrow x_1 \downarrow x_2 \cdots \downarrow x_n) = \sum_{i+j=n+1} m'_i m'_j(\downarrow x_1 \downarrow x_2 \cdots \downarrow x_n)

Proof of Lemma 2.2. $m'_1(x) = (-1)^0 \downarrow \circ m_1 \circ (\downarrow x) = \downarrow m_1(x)$ for any $x$.

Now let $n \geq 2$. The majority of the work here is centered around computing the signs associated with the graded setting. The elements $x_i$ and the maps $\uparrow, \downarrow$, and $m_n$ all contribute to an overall sign via their degrees. Observing these signs, we find

$$m'_n(\downarrow v_1 \downarrow w^{\otimes k} \downarrow v_1 \downarrow w^{\otimes (n-2)-k}), 0 \leq k \leq n - 2:
$$

Case 1: $n$ is even, $k$ is even. Then

$$m'_n(\downarrow v_1 \downarrow v_1 \downarrow v_1 \downarrow w^{\otimes k} \downarrow v_1 \downarrow w^{\otimes (n-2)-k}) = (-1)^{\left\lfloor \frac{n+1}{2} \right\rfloor} \downarrow m_n(\downarrow v_1 \downarrow w^{\otimes k} \downarrow v_1 \downarrow w^{\otimes (n-2)-k})
$$

$$= (-1)^{\left\lfloor \frac{n+1}{2} \right\rfloor} (-1)^{\left\lfloor \frac{n+1}{2} \right\rfloor} \downarrow v_1
$$

If $\frac{n}{2}$ is even, then $(\ast) = (-1)^{\text{even}} (-1)^{\text{odd}} \downarrow v_1 = \downarrow v_1.

If $\frac{n}{2}$ is odd, then $(\ast) = (-1)^{\text{odd}} (-1)^{\text{odd}} \downarrow v_1 = \downarrow v_1.

Case 2: $n$ is even, $k$ is odd. Then

$$m'_n(\downarrow v_1 \downarrow v_1 \downarrow v_1 \downarrow w^{\otimes k} \downarrow v_1 \downarrow w^{\otimes (n-2)-k}) = (-1)^{2\left\lfloor \frac{n+1}{2} \right\rfloor} \downarrow m_n(\downarrow v_1 \downarrow w^{\otimes k} \downarrow v_1 \downarrow w^{\otimes (n-2)-k})
$$

$$= (-1)^{\frac{n+1}{2} - 2} \downarrow v_1
$$

If $\frac{n}{2}$ is even, then $(***) = (-1)^{\text{even}} (-1)^{\text{odd}} \downarrow v_1 = \downarrow v_1.

If $\frac{n}{2}$ is odd, then $(***) = (-1)^{\text{odd}} (-1)^{\text{odd}} \downarrow v_1 = \downarrow v_1.$
Case 3: \( n \) is odd, \( k \) is even. Then

\[
m'_n(\downarrow v_1 \otimes w^{\oplus k} \downarrow v_1 \otimes w^{\oplus(n-2)-k}) = (-1)^{|v_1|+(\frac{n-1}{2}-1)|w|} m_n(v_1 \otimes w^{\oplus k} \otimes v_1 \otimes w^{\oplus(n-2)-k})
\]
\[
= (-1)^{0+\frac{n-1}{2}-1}(-1)^k s_n \downarrow v_1
\]
\[
= (-1)^{\frac{n-1}{2}-1}(-1)^{\frac{(n+1)(n+2)}{2}} \downarrow v_1
\]
\[
= -(-1)^{\frac{n-1}{2}}(-1)^{\frac{(n+1)(n+2)}{2}} \downarrow v_1 \quad (\ast \ast \ast)
\]

If \( \frac{n-1}{2} \) is even, then \((\ast \ast \ast) = -(-1)^{\text{even}}(-1)^{\text{odd*odd}} \downarrow v_1 = \downarrow v_1.
If \( \frac{n-1}{2} \) is odd, then \((\ast \ast \ast) = -(-1)^{\text{odd}}(-1)^{\text{even*odd}} \downarrow v_1 = \downarrow v_1.

Case 4: \( n \) is odd, \( k \) is odd. Then

\[
m'_n(\downarrow v_1 \otimes w^{\oplus k} \downarrow v_1 \otimes w^{\oplus(n-2)-k}) = (-1)^{\frac{n-1}{2}}|w| m_n(v_1 \otimes w^{\oplus k} \otimes v_1 \otimes w^{\oplus(n-2)-k})
\]
\[
= (-1)^{\frac{n-1}{2}}(-1)^k s_n \downarrow v_1
\]
\[
= -(-1)^{\frac{n-1}{2}}(-1)^{\frac{(n+1)(n+2)}{2}} \downarrow v_1
\]
\[
= -(-1)^{\frac{n-1}{2}}(-1)^{\frac{(n+1)(n+2)}{2}} \downarrow v_1 \quad (\ast \ast \ast)
\]

If \( \frac{n-1}{2} \) is even, then \((\ast \ast \ast) = -(-1)^{\text{even}}(-1)^{\text{odd*odd}} \downarrow v_1 = \downarrow v_1.
If \( \frac{n-1}{2} \) is odd, then \((\ast \ast \ast) = -(-1)^{\text{odd}}(-1)^{\text{even*odd}} \downarrow v_1 = \downarrow v_1.

Hence \( m'_n(\downarrow v_1 \otimes w^{\oplus k} \downarrow v_1 \otimes w^{\oplus(n-2)-k}) = \downarrow v_1, \quad 0 \leq k \leq n - 2 \)

Now consider \( m'_n(\downarrow v_1 \otimes w^{\oplus(n-2)} \otimes v_2) \):

Case 1: \( n \) is even. Then

\[
m'_n(\downarrow v_1 \otimes w^{\oplus(n-2)} \otimes v_2) = (-1)^{|v_1|+(\frac{n}{2}-1)|w|} m_n(\downarrow v_1 \otimes w^{\oplus(n-2)} \otimes v_2)
\]
\[
= (-1)^{\frac{n}{2}-1} s_{n+1} \downarrow v_1
\]
\[
= (-1)^{\frac{n}{2}-1}(-1)^{\frac{(n+2)(n+3)}{2}} \downarrow v_1
\]
\[
= (-1)^{\frac{n}{2}-1}(-1)^{\frac{(n+2)(n+3)}{2}} \downarrow v_1 \quad (*)
\]

If \( \frac{n}{2} \) is even, then \((*) = (-1)^{\text{odd}}(-1)^{\text{odd*odd}} \downarrow v_1 = \downarrow v_1.
If \( \frac{n}{2} \) is odd, then \((*) = (-1)^{\text{even}}(-1)^{\text{even*odd}} \downarrow v_1 = \downarrow v_1.
Case 2: \( n \) is odd. Then
\[
m'_{n}(\downarrow v_{1} \otimes w^{\otimes(n-2)} \downarrow v_{2}) = (-1)^{(n-1)/2}m_{n}(\downarrow v_{1} \otimes w^{\otimes(n-2)} \downarrow v_{2})
\]
\[
= (-1)^{n-1\over 2}s_{n+1} \downarrow v_{1}
\]
\[
= (-1)^{n-1\over 2}(-1)^{(n+2)(n+3)\over 2} \downarrow v_{1}
\]
\[
= (-1)^{n-1\over 2}(-1)^{(n+2)(n+3)\over 2} \downarrow v_{1} \quad (**)
\]

If \( n-1 \) is even, then \((**)= (-1)^{\text{even}}(-1)^{\text{odd}}\text{even} \downarrow v_{1}= \downarrow v_{1}\).

If \( n-1 \) is odd, then \((**)= (-1)^{\text{odd}}(-1)^{\text{odd}}\text{odd} \downarrow v_{1}= \downarrow v_{1}\).

Hence \( m'_{n}(\downarrow v_{1} \otimes w^{\otimes(n-2)} \downarrow v_{2})=\downarrow v_{1} \)

The preceding arguments for cases 1 and 2 for \( m'_{n}(\downarrow v_{1} \otimes w^{\otimes(n-2)} \downarrow v_{2}) \) may be repeated for \( m'_{n}(\downarrow v_{1} \otimes w^{\otimes(n-1)}) \).

Thus \( m'_{n}(\downarrow v_{1} \otimes w^{\otimes(n-1)})=\downarrow w \)

\( \square \)

Proof of Lemma 2.4. We first note that
\[
D^{2}(\downarrow x_{1} \otimes \downarrow x_{2} \otimes \cdots \otimes \downarrow x_{n}) = \sum_{i+j \leq n+1} m_{i}m'_{j}(\downarrow x_{1} \otimes \downarrow x_{2} \otimes \cdots \otimes \downarrow x_{n})
\]

since \( m'_{k}(\downarrow x_{1} \otimes \downarrow x_{2} \otimes \cdots \otimes \downarrow x_{l}) = 0 \) for \( k > l \). So

\[
D^{2}(\downarrow x_{1} \otimes \downarrow x_{2} \otimes \cdots \otimes \downarrow x_{n}) = \sum_{i+j \leq n} m_{i}m'_{j}(\downarrow x_{1} \otimes \downarrow x_{2} \otimes \cdots \otimes \downarrow x_{n})
\]
\[
+ \sum_{i+j = n+1} m_{i}m'_{j}(\downarrow x_{1} \otimes \downarrow x_{2} \otimes \cdots \otimes \downarrow x_{n})
\]

Hence it suffices to show that \( \sum_{i+j \leq n} m_{i}m'_{j}(\downarrow x_{1} \otimes \downarrow x_{2} \otimes \cdots \otimes \downarrow x_{n}) = 0 \)

Consider \( \sum_{i+j \leq n} m_{i}m'_{j}(\downarrow x_{1} \otimes \downarrow x_{2} \otimes \cdots \otimes \downarrow x_{n}) \): Since \( i+j \leq n \), we can break this sum up into 4 different types of of elements in \( \downarrow V^{\otimes k} \) based on whether the first and last terms in the tensor product contain \( m_{i} \) or \( m'_{j} \):

- **Type 1**: Elements with first term \( \downarrow x_{1} \) and last term \( \downarrow x_{n} \)
  (example: \( \downarrow x_{1} \otimes \downarrow x_{2} \otimes m'_{1}(\downarrow x_{3}) \otimes m'_{2}(\downarrow x_{4} \otimes \downarrow x_{5}) \otimes \downarrow x_{6} \))
- **Type 2**: Elements with first term \( \downarrow x_{1} \) and last term containing \( m'_{k} \) for some \( k \)
  (example: \( \downarrow x_{1} \otimes \downarrow x_{2} \otimes m'_{3}(\downarrow x_{3} \otimes m'_{2}(\downarrow x_{4} \otimes \downarrow x_{5}) \otimes \downarrow x_{6}) \))
- **Type 3**: Elements with first term containing \( m_{k} \) for some \( k \) and last term \( \downarrow x_{n} \)
(example: \( m'_2 (\downarrow x_1 \otimes \downarrow x_2) \otimes m'_1 (\downarrow x_3 \otimes \downarrow x_4 \otimes \downarrow x_5 \otimes \downarrow x_6) \))

- Type 4: Elements with first term containing \( m'_k \) and last term containing \( m'_l \) for some \( k, l \)
  (example: \( m'_2 (\downarrow x_1 \otimes \downarrow x_2) \otimes \downarrow x_3 \otimes \downarrow x_4 \otimes m'_2 (\downarrow x_5 \otimes \downarrow x_6) \))

Now each term of type 1 must be produced by \( m'_i m'_j \) with \( i + j \leq n - 1 \). Hence, by factorization of tensor products, all possible terms of type 1 are given by:

\[
(-1)^{2|x_1| - 2} \bigg( \downarrow x_1 \otimes \bigg( \sum_{i+j \leq n-1} m'_i m'_j (\downarrow x_2 \otimes \downarrow x_3 \otimes \cdots \otimes \downarrow x_{n-1}) \bigg) \bigg) \otimes \downarrow x_n \\
= \bigg( \downarrow x_1 \otimes (D^2 (\downarrow x_2 \otimes \downarrow x_3 \otimes \cdots \otimes \downarrow x_{n-1})) \bigg) \otimes \downarrow x_n \\
= (\downarrow x_1 \otimes 0) \otimes \downarrow x_n \\
= 0
\]

since \( D^2 = 0 \) when evaluated on \( n - 2 \) terms.

Now since all terms of type 1 form a collection of elements in \( \downarrow V^\otimes k \) that sum up to 0, we can duplicate this collection multiple times. This is significant when we consider all terms of type 2 in conjunction with a set of type 1 terms. Combining all type 2 terms with a set of type 1 terms and factoring tensor products, we get:

\[
(-1)^{2|x_1| - 2} \downarrow x_1 \otimes \bigg( \sum_{i+j \leq n} m'_i m'_j (\downarrow x_2 \otimes \downarrow x_3 \otimes \cdots \otimes \downarrow x_n) \bigg) \\
= \downarrow x_1 \otimes (D^2 (\downarrow x_2 \otimes \downarrow x_3 \otimes \cdots \otimes \downarrow x_n)) \\
= \downarrow x_1 \otimes 0 \\
= 0
\]

since \( D^2 = 0 \) when evaluated on \( n - 1 \) terms.

Hence, all type 2 added together equal 0. All type 3 terms added together equal 0 following a similar argument.

We now consider type 4 terms. Consider an arbitrary element of type 4:

\[ m'_i (\downarrow x_1 \otimes \cdots \otimes \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_{n-j} \otimes m'_j (\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n) \]

Consider how this arbitrary element is generated: We begin with

\[ m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) \]

We then apply \( m'_j \) to the last \( j \) terms, which yields:

\[ (-1)^{|x_1| + \cdots + |x_{n-j}|- (n-j)} m'_i (\downarrow x_1 \otimes \cdots \otimes \downarrow x_{n-j} \otimes m'_j (\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n)) \]
Finally we apply $m_i'$ to the first $i$ terms:

$$(-1)^{|x_1| + \cdots + |x_{n-j}| - (n-j)} m_i'(\downarrow x_1 \otimes \cdots \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_{n-j} \otimes m_j' (\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n) \quad (*)$$

Each of these arbitrary type 4 elements can be paired up with an element generated by $m_j'm_i'$ as follows: Begin with

$$m_j'm_i'(\downarrow x_1 \otimes \cdots \downarrow x_n)$$

Then apply $m_i'$ to the first $i$ terms:

$$m_j'(m_i'(\downarrow x_1 \otimes \cdots \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_n)$$

Finally, apply $m_j'$ to the last $j$ terms:

$$(-1)^{|x_1| + \cdots + |x_{n-j}| - (n-j)+1} m_i'(\downarrow x_1 \otimes \cdots \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_{n-j} \otimes m_j' (\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n) \quad (**)$$

Since these type 4 elements were arbitrary, and $(*) + (**)=0$, all type 4 terms added together equal 0. Hence, all type 1, 2, 3, and 4 terms yield 0, and so

$$\sum_{i+j \leq n} m_i'm_j'(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \downarrow x_n) = 0$$

\[\square\]

Proof of Theorem 2.1. It is clear that each map $m_n$ is of degree $2-n$. To prove that these maps yield an $A_\infty$ structure, one may verify that they satisfy the identity given in definition 1.1. However, this is a rather daunting task, due to the varying signs, $s_n$, accompanying the $m_n$ maps. To utilize an alternative method of proof, we construct a degree 1 coderivation, $D$, as described in section 1.

In the context of Theorem 2.1, we may use the definition for $m'_{k}$ given by Lemma 2.2 to construct $D$. It then suffices to show that $D^2 = 0$.

We aim to prove $D^2 = 0$ by induction on the number of inputs for $D$. It is worth first noting that $D = \sum_{k=1}^{\infty} m'_k$, however $D(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = \sum_{k=1}^{n} m'_k(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$ since $m'_k(\downarrow x_1 \otimes \cdots \downarrow x_n) = 0$ for $k \geq n$.

For $n = 1$, we have $D^2(\downarrow x) = m'_1m'_1(\downarrow x) = m'_1^2(x) = 0 \forall \ x \in V.$
For $n = 2$, we have

\[
D^2(\downarrow x_1, \downarrow x_2) = m_1' m_1'(\downarrow x_1 \downarrow \downarrow x_2) + m_1' m_2'(\downarrow x_1 \downarrow \downarrow x_2)
\]

\[
+ m_2' m_1'(\downarrow x_1 \downarrow x_2) + m_2' m_2'(\downarrow x_1 \downarrow x_2)
\]

\[
= m_1'(m_1'(\downarrow x_1) \downarrow x_2 - (-1)^{|x_1|} x_1 \otimes m_1'(x_2)) + m_1' m_2'(\downarrow x_1 \downarrow x_2)
\]

\[
+ m_2'(m_1'(\downarrow x_1) \downarrow x_2 - (-1)^{|x_1|} x_1 \otimes m_1'(x_2)) + 0
\]

\[
= [m_1' m_1'(\downarrow x_1) \downarrow x_2 + (-1)^{|x_1|} m_1'(\downarrow x_1) \otimes m_1'(\downarrow x_2)]
\]

\[
- (-1)^{|x_1|} [m_1'(x_1) \otimes m_1'(x_2) - (-1)^{|x_1|} x_1 \otimes m_1'(x_2)] + m_1' m_2'(\downarrow x_1 \downarrow x_2)
\]

\[
+ m_2'(m_1'(\downarrow x_1) \downarrow x_2 - (-1)^{|x_1|} m_2'(x_1 \otimes m_1')(x_2))
\]

\[
= m_1' m_2'(\downarrow x_1 \downarrow x_2) + m_2'(m_1'(\downarrow x_1) \downarrow x_2) - (-1)^{|x_1|} m_2'(x_1 \otimes m_1'(x_2))
\]

\[
= 0 \forall x_1, x_2 \in V
\]

Now assume $D^2(\downarrow x_1 \cdots \downarrow \otimes x_{n-1}) = 0$. We aim to show that $D^2(\downarrow x_1 \cdots \downarrow \otimes x_n) = 0$:

By Lemma 2.4, $D^2(\downarrow x_1 \otimes \cdots \otimes x_n) = \sum_{i+j=n+1} m_i' m_j'(\downarrow x_1 \otimes \cdots \otimes x_n)$, hence it suffices to show that $\sum_{i+j=n+1} m_i' m_j'(\downarrow x_1 \otimes \cdots \otimes x_n) = 0$, $\forall x_1 \cdots x_n \in V$.

It is advantageous to approach this problem from the bottom up, since $x_1 \cdots x_n \in V$ implies calculating $3^n$ different combinations of elements. That is, we consider only nontrivial (nonzero) elements in the sum $\sum_{i+j=n+1} m_i' m_j'(\downarrow x_1 \otimes \cdots \otimes x_n)$. Now since $i + j = n + 1$, we observe that $m_i' m_j'(\downarrow x_1 \otimes \cdots \otimes x_n) \in \downarrow V^\otimes 1$. Since, by definition, $m_i'$ cannot produce the element $\downarrow v_2$, the seemingly large task of considering nontrivial $m_i' m_j'(\downarrow x_1 \otimes \cdots \otimes x_n)$ yields only two possibilities:

\[
m_i' m_j'(\downarrow x_1 \otimes \cdots \otimes x_n) = c \downarrow v_1
\]

or

\[
m_i' m_j'(\downarrow x_1 \otimes \cdots \otimes x_n) = c \downarrow w
\]

for some constant, $c$.

**Remark 2.5.** Since production of a $\downarrow v_1$ or $\downarrow w$ relies on the number of $v$’s and $w$’s in the arrangement $\downarrow x_1 \otimes \cdots \otimes x_n$, these possibilities are disjoint.

Therefore if $m_i' m_j'(\downarrow x_1 \otimes \cdots \otimes x_n) \neq 0$ for some $i + j = n + 1$, then $\sum_{i+j=n+1} m_i' m_j'(\downarrow x_1 \otimes \cdots \otimes x_n)$ contains a collection of $\downarrow v_1$’s or $\downarrow w$’s.

We first consider the manner in which $m_i' m_j'(\downarrow x_1 \otimes \cdots \otimes x_n)$ yields a $\downarrow w$:

By definition of $m_n'$, $\downarrow w$ must be produced by $m_i'(\downarrow v_1 \otimes \downarrow w^{\otimes (i-1)})$. Now since a nonzero $m_i'$ will contribute either the $\downarrow v_1$ or a $\downarrow w$ to the arrangement $\downarrow v_1 \otimes \downarrow w^{\otimes (i-1)}$, the
original arrangement $\downarrow x_1 \otimes \cdots \downarrow \otimes x_n$ must contain exactly one more ‘$v$’ ($v = v_1$ or $v_2$), for a total of two $v$’s. It is also worth noting that $x_1 = v_1$, otherwise $m_i m'_j(\downarrow x_1 \otimes \cdots \downarrow \otimes x_n) = 0$.

- **Case 1**: $v = v_1$. Then we have $m_i m'_j(\downarrow v_1 \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k})$, $0 \leq k \leq n - 2$. Now, to produce $(\star)$, $m'_j$ must ‘catch’ (1) both $\downarrow v_1$’s, or (2) only the second $\downarrow v_1$.

(1) We have $m'_j(\downarrow v_1 \downarrow \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) = \downarrow v_1$, $k + 2 \leq j \leq n$. This yields $m'_i(\downarrow v_1 \downarrow w^{\otimes (n-j)}) = \downarrow w$. Now since $k + 2 \leq j \leq n$, there are $n - (k + 2) + 1 = n - k - 1$ such terms in

$$\sum_{i+j=n+1} m'_i m'_j(\downarrow v_1 \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}).$$

(2) We have $(-1)^{|v_1|+\vert k \vert \vert w \vert -(k+1)} m'_i(\downarrow v_1 \downarrow \downarrow w^{\otimes k} \otimes m'_j(\downarrow v_1 \downarrow w^{\otimes (j-1)}) \otimes \downarrow w^{\otimes (n-2)-k-(j-1)}) = - \downarrow w$, $1 \leq j \leq n - k - 1$. Similarly, there are $(n-k-1) - 1 + 1 = n - k - 1$ such terms in

$$\sum_{i+j=n+1} m'_i m'_j(\downarrow v_1 \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}).$$

\[ \Rightarrow \sum_{i+j=n+1} m'_i m'_j(\downarrow v_1 \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) = (n-k-1) \downarrow w - (n-k-1) \downarrow w = 0. \]

- **Case 2**: $v = v_2$. Then we have $m_i' m_j'(\downarrow v_1 \downarrow \downarrow w^{\otimes k} \otimes \downarrow v_2 \otimes \downarrow w^{\otimes (n-2)-k})$, $0 \leq k \leq n - 2$. Similarly, to produce $(\star)$, $m'_j$ must ‘catch’ (1) both $\downarrow v_1$ and $\downarrow v_2$, or (2) only $\downarrow v_2$.

For (1), the only nontrivial way to do this yields:

$$m_{n-k-1}(m'_{k+2}(\downarrow v_1 \downarrow \downarrow w^{\otimes k} \otimes \downarrow v_2) \otimes \downarrow w^{\otimes (n-2)-k}) = \downarrow w$$

and for (2), the only nontrivial way to do this yields:

$$(-1)^{|v_1|+\vert k \vert \vert w \vert -(k+1)} m'_n(\downarrow v_1 \downarrow \downarrow w^{\otimes k} \otimes m'_1(\downarrow v_2) \otimes \downarrow w^{\otimes (n-2)-k}) = - \downarrow w$$

\[ \Rightarrow \sum_{i+j=n+1} m'_i m'_j(\downarrow v_1 \downarrow \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-2)-k}) = \downarrow w - \downarrow w = 0. \]

In either case, if $m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes x_n)$ produces $\downarrow w$’s, then

$$\sum_{i+j=n+1} m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes x_n) = 0.$$

We now consider the manner in which $m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes x_n)$ yields a $\downarrow v_1$:
By definition of \( m'_i \), \( \downarrow v_1 \) must be produced by either \( m'_i(\downarrow v_1 \otimes \downarrow w^\otimes k \otimes \downarrow v_1 \otimes w^\otimes (i-2) - k) \) or \( m'_i(\downarrow v_1 \otimes \downarrow w^\otimes (i-2) \otimes \downarrow v_2) \).

- **Case 1:** \( \downarrow v_1 \) is produced by \( m'_i(\downarrow v_1 \otimes \downarrow w^\otimes k \otimes \downarrow v_1 \otimes \downarrow w^\otimes (i-2) - k) \).

We examine the 4 different possibilities for which \( m'_j \) can yield this arrangement:

(i) \( m'_j \) produces the first \( \downarrow v_1 \). (ii) \( m'_j \) produces a \( \downarrow w \) in \( \downarrow w^\otimes k \).
(iii) \( m'_j \) produces the second \( \downarrow v_1 \). (iv) \( m'_j \) produces a \( \downarrow w \) in \( \downarrow w^\otimes (i-2) - k \).

A key observation to make here is that (i), (ii), (iii), and (iv) imply that the original arrangement \( \downarrow x_1 \otimes \cdots \otimes \downarrow x_n \) must contain exactly 3 \( v \)'s, once again with \( x_1 = v_1 \). This yields 4 subcases:

**Subcase 1:** We have \( m'_i m'_j(\downarrow v_1 \otimes \downarrow w^\otimes k \otimes \downarrow v_1 \otimes \downarrow w^\otimes l \otimes \downarrow v_1 \otimes \downarrow w^\otimes n - k - l - 3) \):

(i) \( m'_j \) must take the first two \( \downarrow v_1 \)'s. We have:

\[
m'_i \left( \left[ m'_j(\downarrow v_1 \otimes \downarrow w^\otimes k \otimes \downarrow v_1 \otimes \downarrow w^\otimes l - (j - k - 2)) \otimes \downarrow v_1 \otimes \downarrow w^\otimes n - k - l - 3 \right] \right) = \downarrow v_1
\]

Now \( k + 2 \leq j \leq l + k + 2 \), so there are \((l + k + 2) - (k + 2) + 1 = l + 1\) such terms.

(ii) \( m'_j \) must take only the second \( \downarrow v_1 \). We have:

\[
(-1)^{|v_1| + |w| - (k + 1)} m'_i \left( \downarrow v_1 \otimes w^\otimes k \otimes \left[ m'_j(\downarrow v_1 \otimes \downarrow w^\otimes (j - 1)) \otimes \downarrow v_1 \otimes \downarrow w^\otimes n - k - l - 3 \right] \right) = - \downarrow v_1
\]

Now \( 1 \leq j \leq l + 1 \), so there are \((l + 1) - 1 + 1 = l + 1\) such terms.

(iii) \( m'_j \) must take the second and third \( \downarrow v_1 \)'s. We have:

\[
(-1)^{|v_1| + |w| - (k + 1)} m'_i \left( \downarrow v_1 \otimes w^\otimes k \otimes \left[ m'_j(\downarrow v_1 \otimes \downarrow w^\otimes j \otimes \downarrow v_1 \otimes \downarrow w^\otimes n - k - j + 1) \right] \right) = - \downarrow v_1
\]

Now \( l + 2 \leq j \leq n - k - 1 \), so there are \((n - k - 1) - (l + 2) + 1 = n - k - l - 2\) such terms.

(iv) \( m'_j \) must take only the third \( \downarrow v_1 \). We have:

\[
(-1)^{|v_1| + (k + 1) |w| - (k + l + 2)} m'_i \left( \downarrow v_1 \otimes w^\otimes k \otimes \downarrow v_1 \otimes \downarrow w^\otimes l \otimes \left[ m'_j(\downarrow v_1 \otimes \downarrow w^\otimes (j - 1)) \otimes \downarrow w^\otimes n - k - l - j - 2 \right] \right) = \downarrow v_1
\]

Now \( 1 \leq j \leq n - k - l - 2 \), so there are \((n - k - l - 2) - 1 + 1 = n - k - l - 2\) such terms.
\[ \sum_{i+j=n+1} m'_i m'_j (\down v_1 \otimes \down w^{\otimes k} \otimes \down v_1 \otimes \down w^{\otimes l} \otimes \down v_1 \otimes \down w^{\otimes n-k-l-3}) = (l+1) \down v_1 - (l+1) \down v_1 - (n-k-l-2) \down v_1 + (n-k-l-2) \down v_1 = 0. \]

\( \circ \) Subcase 2: We have \( m'_i m'_j (\down v_1 \otimes \down w^{\otimes k} \otimes \down v_1 \otimes \down w^{\otimes l} \otimes \down v_2 \otimes \down w^{\otimes n-k-l-3}) \):

By the nature of \( m'_n \), it is advantageous to consider whether or not \( n-k-l-3 = 0 \):

If \( n-k-l-3 = 0 \):

(i) \( m'_j \) must take the first two \( \down v_1 \)'s. We have:

\[ m'_i \left( \left[ m'_j (\down v_1 \otimes \down w^{\otimes k} \otimes \down v_1 \times \down w^{\otimes j-k-2}) \otimes \down w^{\otimes l-(j-k-2)} \right] \otimes \down v_2 \right) = \down v_1 \]

Now \( k+2 \leq j \leq l+k+2 \), so there are \((l+k+2) - (k+2) + 1 = l+1\) such terms.

(ii) \( m'_j \) must take only the second \( \down v_1 \). We have:

\[ (-1)^{|v_1|+|k|+|w|-(k+1)} m'_n (\down v_1 \otimes \down w^{\otimes k} \otimes \left[ m'_j (\down v_1 \otimes \down w^{\otimes (j-1)} \otimes \down w^{\otimes l-(j-1)}) \right] \otimes \down v_2) = - \down v_1 \]

Now \( 1 \leq j \leq l+1 \), so there are \((l+1) - 1 + 1 = l+1\) such terms.

(iii) \( m'_j \) must take the second \( \down v_1 \) and \( \down v_2 \). The only nontrivial way to do this is:

\[ (-1)^{|v_1|+|k|+|w|-(k+1)} m'_{n-l-1} \left( \down v_1 \otimes \down w^{\otimes k} \otimes \left[ m'_{j+2} (\down v_1 \otimes \down w^{\otimes l} \otimes \down v_2) \right] \right) = - \down v_1 \]

(iv) \( m'_j \) must take only \( \down v_2 \). We have:

\[ (-1)^{2|v_1|+(k+l)|w|-(k+l+2)} m'_n \left( \down v_1 \otimes \down w^{\otimes k} \otimes \down v_1 \otimes \down w^{\otimes l} \otimes m'_i (\down v_2) \right) = \down v_1 \]

\[ \Rightarrow \sum_{i+j=n+1} m'_i m'_j (\down v_1 \otimes \down w^{\otimes k} \otimes \down v_1 \otimes \down w^{\otimes l} \otimes \down v_2) = (l+1) \down v_1 - (l+1) \down v_1 - \down v_1 = 0. \]

If \( n-k-l-3 \neq 0 \):

(i) and (ii) are trivial.

(iii) \( m'_j \) must take the second \( \down v_1 \) and \( \down v_2 \). The only nontrivial way to do this is:

\[ (-1)^{|v_1|+|k|+|w|-(k+1)} m'_{n-l-1} \left( \down v_1 \otimes \down w^{\otimes k} \otimes \left[ m'_{l+2} (\down v_1 \otimes \down w^{\otimes l} \otimes \down v_2) \right] \otimes \down w^{\otimes n-k-l-3} \right) = - \down v_1 \]

(iv) \( m'_j \) must take only \( \down v_2 \). We have:

\[ (-1)^{2|v_1|+(k+l)|w|-(k+l+2)} m'_n \left( \down v_1 \otimes \down w^{\otimes k} \otimes \down v_1 \otimes \down w^{\otimes l} \otimes \left[ m'_i (\down v_2) \right] \otimes \down w^{\otimes n-k-l-3} \right) = \down v_1 \]

\[ \Rightarrow \sum_{i+j=n+1} m'_i m'_j (\down v_1 \otimes \down w^{\otimes k} \otimes \down v_1 \otimes \down w^{\otimes l} \otimes \down v_2 \otimes \down w^{\otimes n-k-l-3}) = - \down v_1 + \down v_1 = 0. \]
Subcase 3: We have \( m'_1 m'_2 (\downarrow v_1 \otimes \downarrow w^k \otimes \downarrow v_2 \otimes \downarrow w^l \otimes \downarrow v_1 \otimes \downarrow w^{n-k-l-3}) \):

(i) \( m'_j \) must take the first \( \downarrow v_1 \) and \( \downarrow v_2 \). The only nontrivial way to do this is:

\[
m'_{n-k-1}(\left[ m'_{k+2}(\downarrow v_1 \otimes \downarrow w^k \otimes \downarrow v_2) \right] \otimes \downarrow w^l \otimes \downarrow v_1 \otimes \downarrow w^{n-k-l-3}) = \downarrow v_1
\]

(ii) \( m'_j \) must take \( \downarrow v_2 \) only. The only nontrivial way to do this is:

\[
(-1)^{\left| v_1 + k \right| w - (k+1)} m'_n \left( \downarrow v_1 \otimes w^k \otimes \left[ m'_1(\downarrow v_2) \right] \otimes \downarrow w^l \otimes \downarrow v_1 \otimes \downarrow w^{n-k-l-3}) = - \downarrow v_1
\]

Now (iii) and (iv) are trivial.

\[
\Rightarrow \sum_{i+j=n+1} m'_1 m'_2 (\downarrow v_1 \otimes \downarrow w^k \otimes \downarrow v_2 \otimes \downarrow w^l \otimes \downarrow v_1 \otimes \downarrow w^{n-k-l-3}) = \downarrow v_1 - \downarrow v_1 = 0.
\]

Subcase 4: We have \( m'_1 m'_2 (\downarrow v_1 \otimes \downarrow w^k \otimes \downarrow v_2 \otimes \downarrow w^l \otimes \downarrow v_2 \otimes \downarrow w^{n-k-l-3}) \):

If \( n - k - l - 3 \neq 0 \), then this is trivial. Assume \( n - k - l - 3 = 0 \).

(i) \( m'_j \) must take the first \( \downarrow v_1 \) and first \( \downarrow v_2 \). The only nontrivial way to do this is:

\[
m'_{n-k-1}(\left[ m'_{k+2}(\downarrow v_1 \otimes \downarrow w^k \otimes \downarrow v_2) \right] \otimes \downarrow w^l \otimes \downarrow v_1) = \downarrow v_1
\]

(ii) \( m'_j \) must take second \( \downarrow v_2 \) only. The only nontrivial way to do this is:

\[
(-1)^{\left| v_1 + k \right| w - (k+1)} m'_n \left( \downarrow v_1 \otimes w^k \otimes \left[ m'_1(\downarrow v_2) \right] \otimes \downarrow w^l \otimes \downarrow v_2) = - \downarrow v_1
\]

Now (iii) and (iv) are trivial.

\[
\Rightarrow \sum_{i+j=n+1} m'_1 m'_2 (\downarrow v_1 \otimes \downarrow w^k \otimes \downarrow v_2 \otimes \downarrow w^l \otimes \downarrow v_2 \otimes \downarrow w^{n-k-l-3}) = \downarrow v_1 - \downarrow v_1 = 0.
\]

Hence, our result holds for case 1.

Case 2: \( \downarrow v_1 \) is produced by \( m'_i (\downarrow v_1 \otimes \downarrow w^{(i-2)} \otimes \downarrow v_2) \).

We examine the 2 different possibilities for which \( m'_j \) can yield this arrangement:

(i) \( m'_j \) produces the \( \downarrow v_1 \).
(ii) \( m'_j \) produces a \( \downarrow w \) in \( \downarrow w^{(i-2)} \).
A similar observation to case 1 can be made here regarding the original arrangement $\downarrow x_1 \otimes \cdots \otimes \downarrow x_n$ containing exactly $3$ $v$’s, once again with $x_1 = v_1$. In this case, $x_n = v_2$. This yields $2$ subcases:

- **Subcase 1:** We have $m'_i m'_j (\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-k-3)} \otimes \downarrow v_2)$
  - $(i)$ $m'_j$ must take both $\downarrow v_1$’s. We have:
    \[ m'_i \left( m'_j (\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (j-k-2)} \otimes \downarrow v_2) \right) = \downarrow v_1 \]
    Now $k + 2 \leq j \leq n - 1$, so there are $(n - 1) - (k + 2) + 1 = n - k - 2$ such terms.

  - $(ii)$ $m'_j$ must take the second $\downarrow v_1$ only. We have:
    \[ (-1)^{|v_1|+|k|-(k+1)} m'_i \left( \downarrow v_1 \otimes w^{\otimes k} \otimes m'_j (\downarrow v_1 \otimes \downarrow w^{\otimes j-1} \otimes \downarrow w^{\otimes n-j-2} \otimes \downarrow v_2) \right) = - \downarrow v_1 \]
    Now $1 \leq j \leq n - k - 2$, so there are $(n - k - 2) - (1) + 1 = n - k - 2$ such terms. This implies that
    \[ \sum_{i+j=n+1} m'_i m'_j (\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_1 \otimes \downarrow w^{\otimes (n-k-3)} \otimes \downarrow v_2) = (n-k-2) \downarrow v_1 - (n-k-2) \downarrow v_1 = 0. \]

- **Subcase 2:** We have $m'_i m'_j (\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_2 \otimes \downarrow w^{\otimes (n-k-3)} \otimes \downarrow v_2)$
  - $(i)$ $m'_j$ must take $\downarrow v_1$ and the first $\downarrow v_2$. The only nontrivial way to do this is:
    \[ m'_{n-k-1} \left( m'_{k+2} (\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_2) \right) \otimes \downarrow w^{\otimes n-k-3} \otimes \downarrow v_2) = \downarrow v_1 \]
  - $(ii)$ $m'_j$ must take second $\downarrow v_2$ only. The only nontrivial way to do this is:
    \[ (-1)^{|v_1|+|k|-(k+1)} m'_i (\downarrow v_1 \otimes w^{\otimes k} \otimes m'_j (\downarrow v_2) \otimes \downarrow w^{\otimes n-k-3} \otimes \downarrow v_2) = - \downarrow v_1 \]
    Now $(iii)$ and $(iv)$ are trivial.

\[ \Rightarrow \sum_{i+j=n+1} m'_i m'_j (\downarrow v_1 \otimes \downarrow w^{\otimes k} \otimes \downarrow v_2 \otimes \downarrow w^{\otimes (n-k-3)} \otimes \downarrow v_2) = \downarrow v_1 - \downarrow v_1 = 0. \]

Hence, our result holds for case 2.

So \[ \sum_{i+j=n+1} m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = 0, \forall x_1 \cdots x_n \in V. \]

Thus $D^2 (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = 0$

By induction, $D^2 = 0$ on any number of inputs.
Hence the preceding maps $m_n$ on the graded vector space $V$ form an $A_\infty$ algebra. □

3. **Induced $L_\infty$ Algebra**

The $A_\infty$ algebra structure on $V = V_0 \oplus V_1$ that was constructed in this note may be skew symmetrized to yield an $L_\infty$ algebra structure on $V$; see [5] for details. This $L_\infty$ algebra will thus join the collection of previously defined such structures on $V$. The relationship among these algebras will be a topic for future research.

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