MAJOR/ST 706 - NONLINEAR PROGRAMMING

SOLUTIONS TO PROBLEM C

(PREPARED BY KARTIK)

(2) We have $p_0, p_1, \ldots, p_r \in \mathbb{R}^n$

Consider $\sum_{i=1}^{l} x_i p_i = 0$ where $x_1, x_2, \ldots, x_l \in \mathbb{R}$

$\Rightarrow x_0 p_0 + x_1 p_1 + \ldots + x_l p_l = 0$

$\Rightarrow x_0 A p_0 + x_1 A p_1 + \ldots + x_l A p_l = 0 \Rightarrow \{1\}$

Premultiplying both sides of $\{1\}$ by $p_i^T$ for any $i=0, 1, 2, \ldots, l$

we have

$x_i p_i^T A p_i = 0$  since $p_i^T A p_j = 0$

$\forall i \neq j$

(definition of conjugate directions)

Now $p_i^T A p_i \neq 0$  since $p_i$ is a non-zero vector

$\Rightarrow x_i = 0$

$\Rightarrow \sum_{i=1}^{l} x_i p_i = 0 \Rightarrow x_i = 0 \quad i=1, 2, \ldots, l$

Showing that $p_0, p_1, \ldots, p_l$

are linearly independent.
(3) Work it out yourself

(4) \[ \beta_{k+1} = \frac{Df_{k+1}^T (Df_{k+1} - Df_k)}{||Df_k||^2} \]

For a quadratic function
\[ f(x) = \frac{1}{2} x^T A x - b^T x \]

\[ Df_{k+1} = \lambda_{k+1} \quad \text{and} \quad Df_k = \lambda_k \]

where \( \lambda_i = (A x_i - b) \)

Use Theorem 5.3 on page 109 (Equation 5.161)

to show that
\[ \lambda_{k+1}^T \lambda_k = Df_{k+1}^T Df_k = 0 \]

\[ \beta_{k+1} = \frac{Df_{k+1}^T Df_{k+1}}{||Df_k||^2} \]

\[ \frac{Df_{k+1}^T Df_{k+1}}{||Df_k||^2} \]
(6) We have:

\[ H_{k+1} = H_k - \left( \frac{H_k y_k s_k^T + s_k y_k \text{T} H_k}{y_k^T s_k} \right) \]

\[ + \left( 1 + y_k^T H_k y_k \right) \frac{s_k s_k^T}{y_k^T s_k} \quad (6.17) \]

By rewriting

and

\[ B_{k+1} = B_k - B_k s_k s_k^T B_k + \frac{y_k y_k^T}{s_k^T B_k s_k} \]

We will show that \( M_{k+1} B_{k+1} = I \)

(I will leave it as an exercise to show that \( B_{k+1} H_{k+1} = I \)) \( \therefore B_{k+1} = H_{k+1} \)
\[- \left( \frac{y_k y_k T + s_k y_k^T}{y_k^T s_k} \right) \left( \frac{y_k y_k T}{y_k^T s_k} \right) \]

\[+ \left( 1 + \frac{y_k y_k y_k}{y_k^T s_k} \right) \frac{s_k s_k^T b_k}{y_k^T s_k} - \left( 1 + \frac{y_k y_k y_k}{y_k^T s_k} \right) \frac{s_k}{s_k^T b_k} \left( \frac{s_k^T b_k}{s_k^T s_k} \right) \]

\[+ \left( 1 + \frac{y_k y_k y_k}{y_k^T s_k} \right) \frac{s_k}{s_k^T s_k} \left( \frac{s_k^T y_k}{y_k^T s_k} \right) \]

\[= I - \frac{s_k s_k^T b_k}{s_k^T s_k} + \frac{y_k y_k y_k}{y_k^T s_k} - \left( \frac{y_k y_k s_k^T b_k}{y_k^T s_k} \right) \]

\[- \left( \frac{s_k y_k^T}{y_k^T s_k} \right) + \frac{y_k y_k (s_k^T b_k s_k)}{y_k^T s_k} + \frac{y_k y_k (s_k^T b_k)}{y_k^T s_k} \]

\[- \frac{y_k y_k (s_k^T y_k)}{y_k^T s_k} - \frac{s_k (y_k y_k y_k)}{y_k^T s_k} \]

\[+ \frac{s_k s_k^T b_k}{y_k^T s_k} + \frac{(y_k y_k y_k) s_k s_k^T b_k}{y_k^T s_k} - \frac{s_k s_k^T b_k}{y_k^T s_k} \]

\[- \left( \frac{y_k y_k y_k}{(y_k^T s_k)^2} \right) s_k^T b_k + \frac{s_k y_k^T}{(y_k^T s_k)^2} + \frac{y_k y_k y_k}{(y_k^T s_k)^2} s_k y_k^T \]
Since the remaining terms cancel out as follows:

1 AND 7
2 AND 8
3 AND 6
4 AND 14
5 AND 15
6 AND 12
7 AND 13
Consider
\[ B_{k+1} = B_k + \left( \frac{(y_k-B_k s_k) (y_k-B_k s_k)^T}{(y_k-B_k s_k)^T s_k} \right) \]

Using
\[ (A+ab) = A^{-1} - A^{-1}abA^{-1} \quad (A.27) \]
\[ \frac{1}{1+b^T A^{-1} a} \]

where
\[ A = B_k \quad d = \frac{1}{(y_k-B_k s_k)} \]
\[ (y_k-B_k s_k)^T s_k \]

and \[ b = (y_k-B_k s_k) \]

We have
\[ B_{k+1} = B_k^{-1} - B_k^{-1} (y_k-B_k s_k) (y_k-B_k s_k)^T B_k^{-1} \]
\[ \frac{1}{(y_k-B_k s_k)^T s_k} \]
\[ 1 + \frac{(y_k-B_k s_k)^T B_k^{-1} (y_k-B_k s_k)}{(y_k-B_k s_k)^T s_k} \]
\[ \frac{(y_k-B_k s_k)^T s_k + (y_k-B_k s_k)^T B_k^{-1} (y_k-B_k s_k)}{(y_k-B_k s_k)^T s_k} \]
\[ H_k - (n_k y_k - s_k)(n_k y_k - s_k)^T \]
\[ \frac{y_k^T s_k - s_k^T B_k s_k + (y_k - B_k s_k)^T (n_k y_k - s_k)}{y_k^T s_k - s_k^T B_k s_k + y_k^T y_k - y_k^T s_k} \]
\[ = H_k + (s_k - n_k y_k)(s_k - n_k y_k)^T \]
\[ (s_k - n_k y_k)^T y_k \]