INSTRUCTIONS

Due in class on February 8, 2007. No late homeworks will be accepted without prior instructor approval. All problems are from Chvátal unless otherwise specified.


2. Find conditions on $s$ and $t$ to make the LP problem

\[
\begin{align*}
\text{min} & \quad x_1 + x_2 \\
\text{s.t.} & \quad sx_1 + tx_2 \geq 1 \\
& \quad x_1 \geq 0 \\
& \quad x_2 \text{ is free}
\end{align*}
\]

(a) infeasible, 
(b) feasible, 
(c) have a unique optimal solution, 
(d) have multiple optimal solutions, 
(e) unbounded.

3. Problem 3.9, Part (a), Page 44. Sketch the feasible region of linear program and circle the solutions obtained in the two phases of the simplex method here. Briefly summarize your findings.

4. Consider the following problem

\[
\begin{align*}
\text{max} & \quad 2x_1 + 3x_2 \\
\text{s.t.} & \quad x_1 + 2x_2 \leq 10 \\
& \quad -x_1 + 2x_2 \leq 6 \\
& \quad x_1 + x_2 \leq 6 \\
& \quad x_1, x_2 \geq 0.
\end{align*}
\]

(a) Draw the feasible region of the linear program in $(x_1, x_2)$ space and label the constraints.

(b) Notice that including non-negativity, we have five constraints. What is the solution corresponding to each extreme point of the feasible region?
(c) You will notice that one of the extreme points in the feasible region is the intersection of three constraints, and any two of them will uniquely specify that extreme point. Show that there are three ways to do this. Such an extreme point is said to be degenerate.

(d) Solve the problem graphically and verify that the optimal point is a degenerate extreme point.

(e) Solve the problem by the simplex method. How many iterations do you need? Can you explain the discrepancy?

(f) From Part (c), identify the constraint that causes degeneracy and resolve the problem after throwing this constraint away. Note that degeneracy disappears and the same optimal solution is obtained. How many iterations do you need?

(g) Is it true in general that degenerate extreme points can be made nondegenerate by throwing some constraints away without affecting the feasible region?

5. One of the most important problems in the field of statistics is the linear regression problem. Roughly speaking, this problem involves fitting a straight line to statistical data represented by points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) on a graph. If we denote the line by \(y = a + bx\), the objective is to choose the constants \(a\) and \(b\) to provide the best fit according to some criterion. The criterion usually used is the method of least squares, but there are other interesting criteria where linear programming can be used to solve for the optimal value of \(a\), and \(b\) (see Chapter 14 in Chvátal). Formulate linear programming models for this problem under the following criteria:

(a) Minimize the sum of the absolute deviations of the data from the line; that is

\[
\min \sum_{i=1}^{n} |y_i - (a + bx_i)|,
\]

(b) Minimize the maximum absolute deviation; that is

\[
\min \max_{i=1,\ldots,n} |y_i - (a + bx_i)|.
\]

6. A set \(S \subset \mathbb{R}^n\) is convex if for any \(x, y \in S\), and any \(\lambda \in [0, 1]\), we have \(\lambda x + (1 - \lambda)y \in S\) (see pages 262-263 in Chvátal). In other words, a set is convex if the line segment joining any two of its elements is also contained in the set. Consider the LP in standard form

\[
\begin{align*}
\max & \quad c^T x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

Show the following:

(a) The feasible set of this LP is a convex set.

(b) The optimal solution set of this LP is a convex set. Now, can you construct an LP that has exactly two optimal solutions? Why?