INSTRUCTIONS

1. Please write your name and student number clearly on the front page of the exam.

2. This has to be your own work. Cheating on the exam is not tolerated, and will fetch you a zero for the test.

3. TIME LIMIT: 3 hours

4. There are 6 pages and 5 questions on the exam. Each question appears on a different page. Read each question carefully.

5. The exam is worth 126 points of which 6 points are extra credit. The distribution of points is clearly indicated on the exam.

6. Solve each problem in sufficient detail in the space provided. Please use both sides of each page as needed.

7. Write clearly, including all the steps to the final solution. If I can’t read it, you won’t get credit.

8. Sources: Open book and class notes.

9. You may use an electronic calculator on the exam.

10. A useful formula for the inverse of $2 \times 2$ invertible matrices

$$
\begin{bmatrix}
  a & b \\
  c & d \\
\end{bmatrix}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix}
  d & -b \\
  -c & a \\
\end{bmatrix}.
$$
1. [26 points] Consider the following problem

\[
\begin{align*}
\text{max} \quad & Z = 4x_1 + \alpha x_2 + x_3 + \beta x_4 \\
\text{s.t.} \quad & 4x_1 + ax_2 + x_3 + bx_4 \leq 5 \\
& 3x_1 + cx_2 + 2x_3 + dx_4 \leq 4 \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{align*}
\]

where the values of \( \alpha, \beta, a, b, c, \) and \( d \) are unknown. Let \( x_5 \) and \( x_6 \) denote the slack variables for the two functional constraints. Now suppose Kartik has obtained the following dictionary at some iteration of the simplex method:

\[
\begin{align*}
x_2 &= e + fx_1 + gx_3 - x_5 + x_6 \\
x_4 &= h + ix_1 + jx_3 + x_5 - 2x_6 \\
Z &= \theta + kx_1 + lx_3 - x_5 - x_6
\end{align*}
\]

and you are unable to read some of the entries due to Kartik's bad handwriting :)

Compute the following (in that order):

(a) [6 points] Find the values of \( a, b, c, \) and \( d \).
(b) [4 points] Find the values of \( \alpha \) and \( \beta \).
(c) [2 points] Find the values \( e \) and \( h \).
(d) [2 points] What is the objective value \( \theta \)?
(e) [10 points] Find all the remaining missing entries \( f, g, i, j, k, l \) in the dictionary.
(f) [2 points] Is the dictionary optimal? Why or why not?

1. (a) Since \( x_5 \) and \( x_6 \) are slack variables, we have \( a_5 = [0] \) and \( a_6 = [0] \). Also \( c_5 = c_6 = 0 \).

The general form of a dictionary is

\[
\begin{align*}
X_B = A_B^{-1}b - A_B^{-1}A_N x_N \\
Z = c_B^T A_B^{-1}b + (C_N - A_N^T y)^T x_N
\end{align*}
\]

where \( A_B^T y = c_B \).

We have

\[
\begin{align*}
-A_B^{-1} \begin{bmatrix} a_5^T & a_6^T \end{bmatrix} &= -A_B^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix} \\
A_B &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}
\]
1(b) We have
\[ S_5 = c_5 - a_5^T y \]
\[ S_6 = c_6 - a_6^T y \]
\[ \begin{bmatrix} -1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \]
\[ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
\[ C_B = A_B^T y = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \]
\[ x_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \]

1(c) We have
\[ x_B = \begin{bmatrix} e \\ h \end{bmatrix} = A_B^{-1} b = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \]
\[ \begin{bmatrix} e \\ h \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \]

1(d) We have
\[ \theta = C_B^T x_B = \begin{bmatrix} 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3 + 6 = 9 \]

1(e) We have
\[ \begin{bmatrix} 6 & 9 \\ 1 & 9 \end{bmatrix} = -A_B^{-1} \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \]
\[ = - \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & -3 \end{bmatrix} \]

Also
\[ \begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} S_5 \\ S_3 \end{bmatrix} = \begin{bmatrix} c_5 \\ c_3 \end{bmatrix} - \begin{bmatrix} a_5^T \\ a_3^T \end{bmatrix} y = \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \]
\[ = \begin{bmatrix} -3 \\ -2 \end{bmatrix} \]

1(f) Dictionary is optimal
Since all the coeffs of nonbasic variables in the objective function are all non-positive
2. [25 points] Consider the piecewise-linear minimization problem
\[
\min_z \max_{i=1, \ldots, m} (a_i^T x + b_i)
\]  
(1)

where \(x \in \mathbb{R}^n\), \(a_i\) is the \(i\)th row of \(A \in \mathbb{R}^{m \times n}\), and \(b_i\) is the \(i\)th component of \(b \in \mathbb{R}^m\).

(a) [10 points] Formulate the piecewise-linear minimization problem (1) as an LP. You must give the correct dimensions of all the variables to get full credit.

(b) [10 points] Form the dual of the LP that you obtained in part (a). You must again give the correct dimensions of all the dual variables to get full credit.

(c) [5 points] What are the complementary slackness conditions for this primal-dual pair of LPs. (You will NOT get any credit for writing the complementary slackness conditions on pages 62 and 63 of Chvátal for a general LP, so please confine your answer to the primal-dual LPs in parts (a) and (b))

2(a) We have
\[
Z = \min_x \max_{i=1,2, \ldots, m} (a_i^T x + b_i) \\
= \min_{x, t} \max_{i=1,2, \ldots, m} (a_i^T x + b_i) \leq t \\
= \min_{x, t} t \quad \text{s.t.} \quad (a_i^T x + b_i) \leq t \quad i = 1, 2, \ldots, m \\
= \min_{x, t} t \quad \text{s.t.} \quad A x - e^T t \leq -b \quad \text{where} \ e \in \mathbb{R}^m \\
\quad \quad \text{is the all ones vector} \\
\quad \quad \text{and} \ e^T t \leq \text{vector} \\
\quad \quad \text{where} \ x \in \mathbb{R}^n, \ t \in \mathbb{R}
\]

2(b) Dual is
\[
W = \max_y -b^T y \\
\text{s.t.} \quad \begin{bmatrix} A^T & e^T \end{bmatrix} y = \begin{bmatrix} 0 \ 0 \ 0 \end{bmatrix} \\
y \leq 0
\]
Dual is

\[ w = \max b^T y \]
\[ \text{s.t.} \quad A^T y = 0 \quad \text{where} \quad y \in \mathbb{R}^m \]
\[ e^T y = 1 \]
\[ y \geq 0 \]

2(c) Complementary slackness conditions are

\[ y_i (a_i^T x + b_i - t) = 0 \quad i = 1, 2, \ldots, m \]
3. [30 points] Consider the following problem

\[
\begin{align*}
\text{max } \quad Z &= c_1 x_1 + c_2 x_2 \\
\text{s.t. } \quad 2x_1 - x_2 &\leq b_1 \\
&= b_2 \\
&\geq 0.
\end{align*}
\] (2)

Let \( x_3 \) and \( x_4 \) be the slack variables for the two functional constraints. When \( c_1 = 3, c_2 = -2, b_1 = 30, \) and \( b_2 = 10, \) the simplex method yields the following optimal dictionary:

\[
\begin{align*}
x_2 &= 10 - x_3 + 2x_4 \\
x_1 &= 20 - x_3 + x_4 \\
Z &= 40 - x_3 - x_4.
\end{align*}
\]

(a) [5 points] What is the optimal solution for (2)? Is the optimal solution degenerate? Are there multiple optimal solutions? Construct \( B, A_B, \) and \( c_B \) from the optimal dictionary.

(b) [5 points] Construct the optimal dictionary for the dual problem directly from the optimal dictionary for the primal problem. What is the optimal solution for the dual problem?

(c) [10 points] Use sensitivity analysis to find the allowable range for \( c_1 \) and \( c_2 \) so that the previous primal optimal solution is still optimal in the new problem.

(d) [10 points] Use sensitivity analysis to find the allowable range for \( b_1 \) and \( b_2 \) so the previous primal optimal basis is still feasible in the new problem.

3(a) Optimal solution is \( x^* = \begin{pmatrix} 20 \\ 10 \\ 0 \\ 0 \end{pmatrix} \)

Solution is non-degenerate since \( x_B = \begin{pmatrix} 10 \\ 20 \end{pmatrix} \geq (0, 0) \)

Unique optimal solution since all the coefficients of non-basic variables in obj function are all negative

\( B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \)

\( A_B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \)

\( c_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \)

3(b) Optimal dual dictionary is

\[ y_1 = 1 + 4y_4 + y_3 \]
\[ y_2 = -2y_4 - y_3 \]

Dual solution is \( y^* = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \)

\[-w = -40 -10y_4 - 20y_3\]
3(c) Let \( c_B = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix} \)

we have \( A_B^T y = c_B \)

\[
\begin{bmatrix}
-1 & -1 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} =
\begin{bmatrix}
c_2 \\
c_1
\end{bmatrix}
\]

Solving \( y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1+c_2 \\ -c_1-2c_2 \end{bmatrix} \)

Now \( s_N = c_N - A_N^T y \)

\[
-\begin{bmatrix}
10 \\
01
\end{bmatrix}
\begin{bmatrix}
c_1+c_2 \\
-c_1-2c_2
\end{bmatrix} = \begin{bmatrix}
-c_1-c_2 \\
c_1+2c_2
\end{bmatrix}
\]

Previous solution is still optimal as long as \( s_N \leq 0 \)

i.e \(-c_1-c_2 \leq 0 \) and \( c_1+2c_2 \leq 0 \)

i.e as long as \( -c_2 \leq c_1 \leq -2c_2 \)

3(d) Let \( b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \)

we have \( x_B = A_B^{-1} b = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \)

\[
\begin{bmatrix}
b_1-2b_2 \\
b_1-b_2
\end{bmatrix}
\]

Previous optimal basis is still feasible as long as \( x_B > 0 \)

i.e \( b_1-2b_2 > 0 \) and \( b_1-b_2 > 0 \)

i.e \( b_1 > \), Max \( \{ b_2, 2b_2 \} \)
4. [25 points] Let $A \in \mathbb{R}^{m \times n}$ be a given matrix. Use LP duality to show the following:

(a) [15 points] If every $x \in \mathbb{R}^n$ satisfying $Ax \geq 0$ and $x \geq 0$ must have $x_1 = 0$; then there exists a vector $y \in \mathbb{R}^m$ satisfying $A^Ty \leq 0$, $y \geq 0$, and $a_1^Ty < 0$ where $a_1$ is the first column of the matrix $A$. (Hint: Consider the LP where one maximizes $x_1$ subject to $Ax \geq 0$ and $x \geq 0$).

(b) [10 points] Show the converse of (a): Suppose there exists a vector $y \in \mathbb{R}^m$ satisfying $A^Ty \leq 0$, $y \geq 0$, and $a_1^Ty < 0$; then every vector $x \in \mathbb{R}^n$ satisfying $Ax \geq 0$ and $x \geq 0$ must have $x_1 = 0$.

4(a) Consider the LP

$$
\begin{align*}
2 = \text{Max } & \quad \mathbf{x}^T \mathbf{1} \\
\text{subject to } & \begin{cases}
\mathbf{1} \geq 0 \\
\mathbf{A} \mathbf{x} \geq \mathbf{0} \\
\mathbf{x} \geq \mathbf{0}
\end{cases}
\end{align*}
$$

Since every $x \in \mathbb{R}^n$ satisfying $Ax \geq 0$ and $x \geq 0$ also has $x_1 = 0$, the above LP is optimal with an optimal objective value of 0.

By duality theory, the dual LP

$$
\begin{align*}
W = \text{Min } & \quad \mathbf{y}^T \mathbf{a}_1 \\
\text{subject to } & \begin{cases}
\mathbf{y} \leq \mathbf{0} \\
\mathbf{A}^T \mathbf{y} \geq \mathbf{1} \\
\mathbf{y} \geq \mathbf{0}
\end{cases}
\end{align*}
$$

is also optimal with an objective value of 0.

- If $y$ satisfying $a_1^Ty \geq 1$, $a_2^Ty \geq 0$, ..., $a_n^Ty \geq 0$ and $y \leq 0$

- If $y$ satisfying $a_1^Ty \leq -1$, $a_2^Ty \leq 0$, ..., $a_n^Ty \leq 0$ and $y \geq 0$ (Replacing $y$ with $-y$)

- If $y$ satisfying $A^Ty \leq 0$, $y \geq 0$, and $a_1^Ty < 0$

4(b) We are given a $y \in \mathbb{R}^m$ satisfying $A^Ty \leq 0$, $y \geq 0$, and $a_1^Ty < 0$

If we have a $y \in \mathbb{R}^m$ satisfying $A^Ty \leq 0$, $y \geq 0$, and $a_1^Ty \leq -1$ (why? 2)
We have a vector \( y \in \mathbb{R}^m \) satisfying \( a_i^T y \geq 0 \), \( y \leq 0 \), and \( a_i^T y > 1 \).

The LP

\[
\begin{align*}
W = \min & \quad d_i^T y \\
\text{s.t.} & \quad a_i^T y \geq \frac{1}{10} \\
& \quad y \leq 0
\end{align*}
\]

is feasible and optimal with a objective value of 0.

By duality theory, the dual LP

\[
Z = \max \ x_1
\]

\[
\text{s.t.} \quad A x \geq 0 \quad \text{with a objective value of 0 too}
\]

is optimal

\[
x_2 \geq 0, x_2 \leq 0
\]

i.e., every \( x \in \mathbb{R}^n \) satisfying \( A x \geq 0 \), \( x \leq 0 \) must also satisfy \( x_1 = 0 \).
5. [20 points] Consider the linear program

\[
\max \quad c^T x \\
\text{s.t.} \quad Ax \leq b \\
\quad \quad \quad \quad Ex \leq d \\
\quad \quad \quad \quad x \geq 0
\]

(3)

where \( c, x \in \mathbb{R}^n \), \( A \in \mathbb{R}^{m \times n} \), \( E \in \mathbb{R}^{q \times n} \), \( b \in \mathbb{R}^m \), and \( d \in \mathbb{R}^q \). Let \( y \in \mathbb{R}^m \) and \( w \in \mathbb{R}^q \) denote the dual variables corresponding to the constraints \( Ax \leq b \) and \( Ex \leq d \) in (3) respectively. Let \( x^* \) be an optimal solution to (3) and let \( (y^*, w^*) \) be an optimal solution to its dual.

(a) [5 points] Write down the optimality conditions (primal feasibility, dual feasibility, and complementary slackness) that are satisfied by \( x^* \) and \( (y^*, w^*) \).

(b) [5 points] Consider the LP

\[
\max \quad (c - ATy^*)^T x \\
\text{s.t.} \quad Ex \leq d \\
\quad \quad \quad \quad x \geq 0.
\]

(4)

Show that \( x^* \) and \( w^* \) are also optimal solutions for (4) and its dual (Hint: Show that \( x^* \) and \( w^* \) satisfy the optimality conditions for (4)).

(c) [5 points] Illustrate the property from part (b) with the optimal solution \( x^* = (2, 1) \) and optimal dual solution \( y^* = \frac{1}{2} \) and \( w^* = (\frac{1}{2}, 0) \) to the LP

\[
\max \quad x_1 \\
\text{s.t.} \quad x_1 + x_2 \leq 3 \\
\quad \quad \quad \quad x_1 - x_2 \leq 1 \\
\quad \quad \quad \quad x_2 \leq 2 \\
\quad \quad \quad \quad x_1, x_2 \geq 0.
\]

(5)

where \( x_1 + x_2 \leq 3 \) is in \( Ax \leq b \), and \( x_1 - x_2 \leq 1 \), \( x_2 \leq 2 \) are in \( Ex \leq d \) respectively. You will solve (3) and (4) for this example graphically.

(d) [5 points] Show that an optimal solution to (4) need not be an optimal solution to (3) by looking at the LP (5). Under what conditions is an optimal solution to (4) also an optimal solution to (3)?

5(a)

\[
\text{Dual to (3) is } \quad \text{min } \quad b^T y + d^T w \\
\text{s.t. } \quad ATy + Ew \geq c \\
\quad \quad \quad \quad y \geq 0, \quad w \geq 0
\]

\[
5 \text{ (a)}
\]
Optimality conditions satisfied by $x^*$ and $(y^*, w^*)$ include:

$$Ax^* \leq b \quad \text{and} \quad Ex^* \leq d$$

$$x^* \geq 0 \quad \text{(Primal Feas)}$$

$$AT^* y^* \geq c^* \quad \text{and} \quad y^* \geq 0 \quad \text{and} \quad w^* \geq 0 \quad \text{(Dual Feas)}$$

$$y^*_i (a_i^T x^* - b_i) = 0 \quad i = 1, 2, \ldots, m$$

$$w^*_j (c_j x^* - d_j) = 0 \quad j = 1, 2, \ldots, q$$

$$x^*_k (AT^* y^* + ET^* w^* - c^*) = 0 \quad k = 1, 2, \ldots, l$$

The dual to (4) is:

$$\text{Min } d^T w$$

$$s + ET w \geq (c - AT y^*)$$

Opt conditions satisfied by an optimal $x^*$ and $w^*$ to (4):

$$Ex^* \leq d \quad ET^* w \geq (c - AT y^*)$$

$$x^* \geq 0 \quad w^* \geq 0$$

$$y^* (a_i^T x^* - b_i) = 0$$

$$w^*_j (c_j x^* - d_j) = 0$$

$$x^*_k (AT^* y^* + ET^* w^* - c^*) = 0$$

which are already satisfied by the optimal $x^*$ and $w^*$ to (3) (from (2))

$x^*$ and $w^*$ are also optimal solutions to (4) and its dual respectively.

The LP (4) for example 5:

$$\text{Max } \frac{1}{2} x_1 - \frac{1}{2} x_2$$

$$s + x_1 - x_2 \leq 1$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

This LP has multiple optimal solutions (line segment joining $(1,0)$ to $(3,2)$)

Moreover, $(2,1)$ is one of these optimal solutions.

An optimal solution to (4) would also be an optimal solution to (3) if and only if $x^*$ is unique.