INSTRUCTIONS

1. Please write your name and student number clearly on the front page of the exam.

2. This has to be your own work. Cheating on the exam is not tolerated and will fetch you a zero for the test.

3. TIME LIMIT: 90 minutes

4. There are 5 pages and 4 questions on the exam. Each question appears on a different page. Read each question carefully.

5. The exam is worth 105 points with 5 extra credit points. The distribution of these points is clearly indicated on the exam.

6. Solve each problem in sufficient detail in the space provided. Please use both sides of each page as needed.

7. Write clearly, including all the steps to the final solution. If I can’t read it, you won’t get credit.

8. This is a closed book exam. You are allowed 2 crib sheets of formulas on the exam.

9. You may also use an electronic calculator on the exam.
1. [35 points] Find the solution to the initial value problem

\[ \frac{d^2y}{dt^2} + 9y = 27; \]
\[ y(0) = 4, \quad \frac{dy}{dt}(0) = 6. \]

(1) **STEP 1:**

Solve auxiliary equation:

\[ x^2 + 9 = 0 \quad x_1 = 3i \quad x_2 = -3i \]

\[ y_1(t) = \cos 3t \quad y_2(t) = \sin 3t \]

\[ \therefore y_g(t) = c_1 y_1(t) + c_2 y_2(t) \]
\[ = c_1 \cos 3t + c_2 \sin 3t \]

(2) **STEP 2:**

Find particular solution \( y_p(t) \)

We have \( y_p(t) = A_0 \)

Plugging \( y_p(t) = A_0 \) into the ODE, we have

\[ 0 + 9A_0 = 27 \quad \therefore A_0 = 3 \]

\[ \therefore \text{Particular solution} \quad y_p(t) = 3 \]

(3) **General solution to our ODE is**

\[ y(t) = y_g(t) + y_p(t) = c_1 \cos 3t + c_2 \sin 3t + 3 \]

We have \( y'(t) = -3c_1 \sin 3t + 3c_2 \cos 3t \)

We have \( 4 = y(10) = c_1 + 3 \quad \therefore c_1 = 1 \)

We have \( 6 = y'(10) = 3c_2 \quad \therefore c_2 = 2 \)

\[ \therefore y(t) = \cos 3t + 2\sin 3t + 3 \]
2. [20 points] Determine the following inverse Laplace transform

\[ \mathcal{L}^{-1} \left\{ \frac{8}{s^{2} - 2s + 1} \right\}. \]

we have

\[ \mathcal{L}^{-1} \left\{ \frac{s}{s^{2} - 2s + 1} \right\} = \mathcal{L}^{-1} \left\{ \frac{s}{(s-1)^{2}} \right\} \]

\[ = \mathcal{L}^{-1} \left\{ \frac{A}{s-1} + \frac{B}{(s-1)^{2}} \right\} \]

\[ = \mathcal{L}^{-1} \left\{ \frac{A(s-1) + B}{(s-1)^{2}} \right\} \]

\[ \therefore \text{we have} \quad A(s-1) + B = s \]

Setting \( s = 1 \) we have

\[ B = 1 \]

\[ \therefore A(s-1) + 1 = s \]

\[ \therefore A(s-1) = (s-1) \]

\[ \therefore A = 1 \]

\[ \therefore \mathcal{L}^{-1} \left\{ \frac{s}{s^{2} - 2s + 1} \right\} = \mathcal{L}^{-1} \left\{ \frac{(s-1) + 1}{(s-1)^{2}} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} \]

\[ = \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^{2}} \right\} \]

\[ = e^{t} + e^{t}t \]

\[ = e^{t}(1 + t) \]
3. [20 points] Recall the following definition

\[
cosh(bt) = \frac{e^{bt} + e^{-bt}}{2}
\]  

for the hyperbolic cosine. Use (1) to compute the Laplace transform of the following functions; \(a\) and \(b\) are real constants.

(a) [5 points] \(\cosh(bt)\).

(b) [5 points] \(\frac{d}{dt} \cosh(bt)\).

(c) [5 points] \(e^{at} \cosh(bt)\).

(d) [5 points] \(t \cosh(bt)\).

\[3(a)\]

\[
L \left\{ \cosh(bt) \right\} = L \left\{ \frac{1}{2} \left( e^{bt} + e^{-bt} \right) \right\}
\]

\[
= \frac{1}{2} \left[ L \left\{ e^{bt} \right\} + L \left\{ e^{-bt} \right\} \right]
\]

\[
= \frac{1}{2} \left[ \frac{1}{s-b} + \frac{1}{s+b} \right]
\]

\[
= \frac{1}{2} \left[ \frac{2s}{s^2-b^2} \right]
\]

\[
= \frac{s}{s^2-b^2} \rightarrow F(s)
\]

\[3(b)\]

\[
L \left\{ \frac{d}{dt} \cosh(bt) \right\} = L \left\{ f'(t) \right\} = sF(s) - f(0)
\]

where

\[
f(t) = \cosh(bt)
\]

\[
= \frac{s^2}{s^2-b^2} - \frac{1}{2} \left[ e^0 + e^{-0} \right]
\]

\[
= \frac{s^2}{s^2-b^2} - 1
\]

\[
= \frac{b^2}{s^2-b^2}
\]
\[3(c) \quad L \{ e^{at} \cosh(bt) \} = L \{ e^{at} f(t) \} = F(s-a) \text{ where } f(t) = \cosh(bt) \]

\[
= \frac{(s-a)}{(s-a)^2 - b^2}
\]

\[3(a) \quad L \{ \cosh(bt) \} = L \{ \frac{f(t)}{s} \} = (-1) \frac{d}{ds} F(s) \text{ where } f(t) = \cosh(bt) = (-1) \frac{d}{ds} \left( \frac{s}{s^2 - b^2} \right) \]

\[
= (-1) \left[ \frac{(s^2 - b^2) - s \cdot 2s}{(s^2 - b^2)^2} \right]
\]

\[
= \frac{s^2 + b^2}{(s^2 - b^2)^2}
\]
4. [30 points] Use Laplace transforms to solve the initial value problem

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0;$$

$$y(0) = -1, \quad \frac{dy}{dt}(0) = 6.$$  \[\rightarrow (1)\]

**Solution:**

Let $$y(s) = L\{y(t)\}$$

**Step 1:** Taking Laplace transforms on both sides of ODE gives

\[
\begin{align*}
(s^2y(s) - sy(0) - y'(0)) + 6(sy(s) - y(0)) + 9y(s) &= 0 \\
(s^2y(s) + sy(s) + 1) + 6y(s) &= 0 \\
(s^2 + 6s + 9)y(s) &= -5
\end{align*}
\]

\[\Rightarrow (2)\]

**Step 2:**

\[y(s) = \frac{-5}{(s^2 + 6s + 9)} = \frac{-5}{(s+3)^2} \rightarrow (2)\]

**Step 3:** Taking inverse Laplace on both sides of (1) gives

\[y(t) = L^{-1}\left\{ \frac{-5}{(s+3)^2} \right\}
= L^{-1}\left\{ \frac{A}{s+3} + \frac{B}{(s+3)^2} \right\}
= L^{-1}\left\{ \frac{A(s+3) + B}{(s+3)^2} \right\}
\]
We have
\[ A(s+3) + B = -s \]
Setting \( s = -3 \), we have
\[ B = 3 \]
\[ A(s+3) + 3 = -s \]
Setting \( s = 0 \), we have
\[ 3A = -3 \quad A = -1 \]
\[ y(t) = L^{-1}\left\{ \frac{-1}{s+3} + \frac{3}{(s+3)^2} \right\} \]
\[ = -L^{-1}\left\{ \frac{1}{s+3} \right\} + 3L^{-1}\left\{ \frac{1}{(s+3)^2} \right\} \]
\[ = -e^{-3t} + 3e^{-3t}t \]
\[ = e^{-3t}(3t-1) \]